TRANSIENT PLANE WAVES IN MULTILAYERED HALF-SPACE

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Abstract: Considered the dynamic problem of the theory of elasticity for multilayered half-space. Boundary surface of inhomogeneous half-space loaded with normal load, and the boundaries of separation layers are in conditions of ideal mechanical contact. The formulation involves non-classical separation of equations of motion using two functions with a particular mechanical meaning volumetric expansion and function of acceleration of the shift. In terms of these functions obtained two wave equation, written boundary conditions and the conditions of ideal mechanical contact of layers. Using the Laguerre and Fourier integral transformations was obtained the solution of the formulated problem. The results of the calculation of the stress-strain state in the half-space with a coating for a local impact loading are presented.

Key words: Dynamical Problem of Elasticity, Multilayered Half-Space, Analytical Solution, Fourier Integral Transformation

1. INTRODUCTION

Questions concerning the propagation of a elastic waves in layered bodies under local loading of their surfaces are urgent due to great number of up to date problems of soil mechanics, acoustic flaw detection, problems involving strength of composite materials etc. In general the analysis of this phenomenon is connected with solving a spatial dynamical elasticity problem with corresponding initial and boundary conditions and joining conditions on flat or curvilinear surfaces. At present there have been formed different approaches to solving dynamical elasticity problems.

Among the analytical methods in the first place there should be mentioned methods connected with the applying of Fourier and Laplace integral transformations (Slyep'yan and Yakovlyev, 1980). However the exact inversion of the received transforms can be performed only in few simplest cases and that's why here various numerical and asymptotical methods must be applied. And it's obvious it influences the accuracy and reliability of the received results.

Among other approaches to solving dynamical elasticity problems the method of characteristics (Chou and Greif, 1968; Yang and Achenbach, 1970), the method of using finite integral transformations (Slyep'yan and Yakovlyev, 1980; Wankhede and Bhonse, 1980), the method of summing up of elementary waves (Bedford and Drumheller, 1994) and others (Pao and Mow, 1973; Poruchikov, 1986) can be pointed out. But in a spatial case and in the case of great number of separation boundaries the mentioned above methods are not always effective.

In this paper Laguerre integral transform with respect to time variable (Galazyuk, 1981) is offered for solving this kind of problems. The advantage of this approach is the simplicity of performing an inversion which consists in summing up an orthogonal series as well as the possibility of construction of simple algorithm of series coefficients exact finding. The application of this method as well as the convergence of all procedures is justified by known theorems from an orthogonal polynomials theory (Szego, 1959). More over the scaling factor, introduced in Laguerre transform, permits us to "stretch" or "contract" time interval to depend on numerical analysis needs and make use of corresponding theorems about limiting value.

2. BASIC EQUATIONS

Let us a medium which consists of M flat layers with different thicknesses and different physical-mechanical properties. Each i-th layer has thickness h_i, $1 \le i \le M$ (number M denotes the half-space) and is characterised by Lame elastic constants λ_i, μ_i and density ρ_i. When body forces applied to the medium are identity equal to zero then deformational field for every i-th layer can be found from equation of motion:

$$c_{1,i}^2\, grad(div\mathbf{U}^{(i)}) - c_{2,i}^2 rot(rot\mathbf{U}^{(i)}) = \partial_t^2 \mathbf{U}^{(i)} \qquad (2.1)$$

with corresponding initial and boundary conditions.

The same as in Galazyuk and Chumak (1991), vector equations (2.1) in mixed cylindrical coordinates α, β, γ referred to some linear dimension h with the help volumetric expansion:

$$\theta^{(i)} = h(CD)^{-1}[\partial_\alpha(Du^{(i)}) + \partial_\beta(Cv^{(i)})] + \partial_\gamma w^{(i)} \qquad (2.2)$$

twist functions:

$$\chi^{(i)} = h(CD)^{-1}[\partial_\alpha(Dv^{(i)}) - \partial_\beta(Cu^{(i)})] \qquad (2.3)$$

and functions of acceleration of a shift:

$$\varphi^{(i)} = \tilde{\eta}_i^2 \partial_\tau^2 \omega^{(i)} - \partial_\gamma \theta^{(i)} \qquad (2.4)$$

can be divided into 3M scalar wave equations:

$$\frac{h}{CD}\left[\partial_\alpha\left(\frac{D}{C}\partial_\alpha\theta^{(i)}\right) + \partial_\beta\left(\frac{C}{D}\partial_\beta\theta^{(i)}\right)\right] + \partial_\gamma^2\theta^{(i)} = \tilde{\eta}_i^2\partial_\tau^2\theta^{(i)}$$

$$\frac{h}{CD}\left[\partial_\alpha\left(\frac{D}{C}\partial_\alpha\chi^{(i)}\right)+\partial_\beta\left(\frac{C}{D}\partial_\beta\chi^{(i)}\right)\right]+\partial_\gamma^2\chi^{(i)}=\tilde{\eta}_i^2\eta_i^2\partial_\tau^2\chi^{(i)}$$

$$\frac{h}{CD}\left[\partial_\alpha\left(\frac{D}{C}\partial_\alpha\varphi^{(i)}\right)+\partial_\beta\left(\frac{C}{D}\partial_\beta\varphi^{(i)}\right)\right]+\partial_\gamma^2\varphi^{(i)}=\tilde{\eta}_i^2\eta_i^2\partial_\tau^2\varphi^{(i)}$$

Here and in the following α,β – dimensionless orthogonal coordinates on the plane $\gamma=0$, $hu^{(i)}, hv^{(i)}, hw^{(i)}$ – components of the displacement vector $\mathbf{U}^{(i)}$; $\tau=\frac{c_{1,0}t}{h}$ is the dynamical time; $\eta_i=\frac{c_{1,i}}{c_{2,i}}$, $\tilde{\eta}_i=\frac{c_{1,0}}{c_{1,i}}$, $c_{1,i}=\sqrt{\frac{\lambda_i+2\mu_i}{\rho_i}}$, $c_{2,i}=\sqrt{\frac{\mu_i}{\rho_i}}$ – are the longitudinal and transversal waves propagation velocities in the material of i-th layer; $c_1,0$ – velocity of wave propagation in some medium (it is selected to depend from a numerical analysis tasks); $C(\alpha,\beta)=h\sqrt{E}$, $D(\alpha,\beta)=h\sqrt{G}$, $E(\alpha,\beta)$, $G(\alpha,\beta)$ – coefficients of the first quadric form in the system α,β.

In the following we suppose that the transient processes source in the initially immovable layered medium is the shock loading of its boundary $\gamma=0$. This loading in turn comes to action of a symmetrical to γ-axis normal loading in the circle region of α_0 radius. On the joining surfaces conditions of perfect mechanical contact take place and on infinity displacements and stresses are absent.

In the considered statement our problem of determination of stress-strain state is plane and that is why all the characteristics of the problem will be functions only of α,β,γ and function $\chi^{(i)}$ introduced by (2.3) is identically equal to zero. In the dimensionless rectangular coordinates $(\alpha\to x,\ \beta\to y, \gamma\to z)$ coefficients of the first quadric form will be $C(x,y)=h$, $D(x,y)=h$.

Then in terms of separation functions $\theta^{(i)}(x,z,\tau)$ and $\varphi^{(i)}(x,z,\tau)$ problem reduces to initial- boundary problem of mathematical physics for the system of $2M$ wave equations:

$$\partial_x^2\theta^{(i)}+\partial_z^2\theta^{(i)}=\tilde{\eta}_i^2\partial_\tau^2\theta^{(i)}\ ,\ 1\le i\le M \tag{2.5}$$

$$\partial_x^2\varphi^{(i)}+\partial_z^2\varphi^{(i)}=\tilde{\eta}_i^2\eta_i^2\partial_\tau^2\varphi^{(i)}\ ,\ 1\le i\le M \tag{2.6}$$

under zero initial conditions:

$$\theta^{(i)}=\partial_\tau\theta^{(i)}=\varphi^{(i)}=\partial_\tau\varphi^{(i)}=0 \tag{2.7}$$

boundary conditions on the surface $z=0$:

$$\left(\frac{\eta_1^2}{2}-1\right)\partial_\tau^2\theta^{(1)}+\frac{1}{\tilde{\eta}_1^2}\left(\partial_z^2\theta^{(1)}+\partial_z\varphi^{(1)}\right)=\frac{\eta_1^2}{2}\partial_\tau^2 p_z\,, \tag{2.8}$$

$$\frac{1}{\tilde{\eta}_1^2}\partial_x^2\left[\partial_z\theta^{(1)}+\varphi^{(1)}\right]-\frac{\eta_1^2}{2}\partial_\tau^2\varphi^{(1)}=0\ ; \tag{2.9}$$

Joining conditions on the surfaces $z=z_i=\frac{h_i}{h}$:

$$\left(\frac{\eta_i^2}{2}-1\right)\partial_\tau^2\theta^{(i)}+\frac{1}{\tilde{\eta}_i^2}\left(\partial_z^2\theta^{(i)}+\partial_z\varphi^{(i)}\right)=\Omega_i\left\{\left(\frac{\eta_{i+1}^2}{2}-1\right)\partial_\tau^2\theta^{(i+1)}+\frac{1}{\tilde{\eta}_{i+1}^2}\left(\partial_z^2\theta^{(i+1)}+\partial_z\varphi^{(i+1)}\right)\right\} \tag{2.10}$$

$$\frac{1}{\tilde{\eta}_i^2}r\partial_x^2\left[\partial_z\theta^{(i)}+\varphi^{(i)}\right]-\frac{\eta_i^2}{2}\partial_\tau^2\varphi^{(i)}=\Omega_i\left\{\frac{1}{\tilde{\eta}_{i+1}^2}\partial_x^2\left[\partial_z\theta^{(i+1)}+\varphi^{(i+1)}\right]-\frac{\eta_{i+1}^2}{2}\partial_\tau^2\varphi^{(i+1)}\right\} \tag{2.11}$$

$$\frac{1}{\tilde{\eta}_i^2}\left[\partial_z\theta^{(i)}+\varphi^{(i)}\right]=\frac{1}{\tilde{\eta}_{i+1}^2}\left[\partial_z\theta^{(i+1)}+\varphi^{(i+1)}\right]; \tag{2.12}$$

$$\frac{1}{\tilde{\eta}_i^2}[\partial_x^2\theta^{(i)}-\partial_z\varphi^{(i)}]=\frac{1}{\tilde{\eta}_{i+1}^2}[\partial_x^2\theta^{(i+1)}-\partial_z\varphi^{(i+1)}] \tag{2.13}$$

and condition on the infinity:

$$\theta^{(M)}=\varphi^{(M)}=0,\ z\to\infty \tag{2.14}$$

In the equations (2.5)-(2.13) $\Omega_i=\frac{\rho_{i+1}c_{2,i+1}^2}{\rho_i c_{2,i}^2}$, $p_z=\frac{p(x,\tau)}{\rho_1 c_{1,1}^2}$, $p(x,\tau)$ – known normal force loading.

Using the functions $\theta^{(i)}$ and $\varphi^{(i)}$, $1\le i\le M$ found from the problem (2.5)-(2.14) normal components of displacement vector $w^{(i)}(r,z,\tau)$ can be determined as the solutions of Cauchy problems:

$$\partial_\tau^2 w^{(i)}=\frac{1}{\tilde{\eta}_i^2}\left[\varphi^{(i)}+\partial_z\theta^{(i)}\right],\ w^{(i)}=\partial_\tau w^{(i)}=0,\ \tau=0 \tag{2.15}$$

and radial components $u^{(i)}(r,z,\tau)$ can be determined with the help of integrals:

$$u^{(i)}(x,z,\tau)=\int_0^x[\theta^{(i)}(y,z,\tau)-\partial_z w^{(i)}(y,z,\tau)]dy\,\cdot \tag{2.16}$$

3. SOLUTION OF PROBLEM

Laguerre integral transformation can be introduced by the expression:

$$F_n(r,z)=\int_0^\infty e^{-\lambda\tau}F(r,z,\tau)L_n(\lambda\tau)d\tau\,,\ n=0,1,2,\ldots, \tag{3.1}$$

where $\mathrm{L_n}(\lambda\tau), \mathrm{n}=0,1,2,\ldots$ – Laguerre polynomials. The orthogonal series:

$$F(x,z,\tau)=\lambda\sum_{n=0}^\infty F_n(x,z)L_n(\lambda\tau) \tag{3.2}$$

serve as the inversion formula for the transformation (3.1).

Under some limitations on the function $F(x,z,\tau)$ (Szego, 1959) the integral (3.1) exists and series (3.2) uniformly coincides whit arbitrary interval $[a,b]$ from $(0,\infty)$.

Applying Fourier integral transformation with respect to spatial coordinate x and transformation (3.1) with respect to τ and using then zero initial conditions (2.6) and differentiation formulae:

$$\partial_\tau^2\left[\exp(-\lambda\tau)L_n(\lambda\tau)\right]=-\lambda^2\exp(-\lambda\tau)\sum_{m=0}^n(n-m+1)L_m(\lambda\tau)\,,$$

we'll receive a triangular sequence of boundary value problems of the following kind:

$$d_z^2\bar{\theta}_n^{(i)} - \varsigma_{1,i}^2\bar{\theta}_n^{(i)} = \omega_{1,i}^2\sum_{m=0}^{n-1}(n-m+1)\bar{\theta}_m^{(i)}\ ,\ 1\le i\le M \tag{3.3}$$

$$d_z^2\bar{\varphi}_n^{(i)} - \varsigma_{2,i}^2\bar{\varphi}_n^{(i)} = \omega_{2,i}^2\sum_{m=0}^{n-1}(n-m+1)\bar{\varphi}_m^{(i)}\ ,\ 1\le i\le M \tag{3.4}$$

$$\left(\frac{\eta_1^2}{2}-1\right)\lambda^2\sum_{m=0}^{n}(n-m+1)\,\bar{\theta}_m^{(1)} + \frac{1}{\tilde{\eta}_1^2}\left(d_z^2\bar{\theta}_n^{(1)} + d_z\bar{\varphi}_n^{(1)}\right) = \frac{\eta_1^2}{2}\lambda^2\sum_{m=0}^{n}(n-m+1)\,\bar{p}_z^m\ , \quad z=0; \tag{3.5}$$

$$\frac{\xi^2}{\tilde{\eta}_1^2}\left[d_z\bar{\theta}_n^{(1)} + \bar{\varphi}_n^{(1)}\right] + \frac{\eta_1^2}{2}\lambda^2\sum_{m=0}^{n}(n-m+1)\,\bar{\varphi}_m^{(1)} = 0\ , \quad z=0 \tag{3.6}$$

$$\left(\frac{\eta_i^2}{2}-1\right)\lambda^2\sum_{m=0}^{n}(n-m+1)\,\bar{\theta}_m^{(i)} + \frac{1}{\tilde{\eta}_i^2}\left(d_z^2\bar{\theta}_n^{(i)} + d_z\bar{\varphi}_n^{(i)}\right) =$$
$$= \Omega_i\left\{\left(\frac{\eta_{i+1}^2}{2}-1\right)\lambda^2\sum_{m=0}^{n}(n-m+1)\,\bar{\theta}_m^{(i+1)} + \frac{1}{\tilde{\eta}_{i+1}^2}\left(d_z^2\bar{\theta}_n^{(i+1)} + d_z\bar{\varphi}_n^{(i+1)}\right)\right\},\quad z=z_i,\quad 1\le i\le M-1; \tag{3.7}$$

$$\frac{\xi^2}{\tilde{\eta}_i^2}\left[d_z\bar{\theta}_n^{(i)} + \bar{\varphi}_n^{(i)}\right] + \frac{\eta_i^2}{2}\lambda^2\sum_{m=0}^{n}(n-m+1)\,\bar{\varphi}_m^{(i)} =$$
$$= \Omega_i\left\{\frac{\xi^2}{\tilde{\eta}_{i+1}^2}\left[d_z\bar{\theta}_n^{(i+1)} + \bar{\varphi}_n^{(i+1)}\right] + \frac{\eta_{i+1}^2}{2}\lambda^2\sum_{m=0}^{n}(n-m+1)\,\bar{\varphi}_m^{(i+1)}\right\},\quad z=z_i,\qquad 1\le i\le M-1; \tag{3.8}$$

$$\frac{1}{\tilde{\eta}_i^2}\left[d_z\bar{\theta}_n^{(i)} + \bar{\varphi}_n^{(i)}\right] = \frac{1}{\tilde{\eta}_{i+1}^2}\left[d_z\bar{\theta}_n^{(i+1)} + \bar{\varphi}_n^{(i+1)}\right],\quad z=z_i\ ,\quad 1\le i\le M-1; \tag{3.9}$$

$$\frac{1}{\tilde{\eta}_i^2}\left[\xi^2\bar{\theta}_n^{(i)} + d_z\bar{\varphi}_n^{(i)}\right] = \frac{1}{\tilde{\eta}_{i+1}^2}\left[\xi^2\bar{\theta}_n^{(i+1)} + d_z\bar{\varphi}_n^{(i+1)}\right],\quad z=z_i,\quad 1\le i\le M-1 \tag{3.10}$$

$$\bar{\theta}_n^{(M)} = \bar{\varphi}_n^{(M)} = 0,\qquad z\to\infty. \tag{3.11}$$

In the expressions (3.3) – (3.11) $n = 0,1,2,\ldots;\ \sum_{n=0}^{-1} \equiv 0$ are the Fourier–Laguerre transforms:

$$\bar{F}_n^{(i)}(\xi,z) = \int_0^\infty rJ_0(\xi r)\int_0^\infty e^{-\lambda\tau}F^{(i)}(r,z,\tau)L_n(\lambda\tau)d\tau dr$$

$$\varsigma_{1,i}^2 = \xi^2 + \lambda^2\tilde{\eta}_i^2;$$

$$\varsigma_{2,i}^2 = \xi^2 + \lambda^2\tilde{\eta}_i^2\eta_i^2;\qquad \omega_{1,i}^2 = \lambda^2\tilde{\eta}_i^2;\ \ \omega_{2,i}^2 = \lambda^2\tilde{\eta}_i^2\eta_i^2$$

As it is known (Galazyuk and Gorekcho, 1983) the general solution of the triangular sequence of ordinary differential equations (3.3), (3.4) can be represented as the algebraically convolution:

$$\bar{\theta}_n^{(i)}(z) = \sum_{j=0}^{n}\left[A_{n-j}^{(i)}G_{1,j}^{(i)}(z) + B_{n-j}^{(i)}W_{1,j}^{(i)}(z)\right] \tag{3.12}$$

$$\bar{\Phi}_n^{(i)}(z) = \sum_{j=0}^{n}\left[C_{n-j}^{(i)}G_{2,j}^{(i)}(z) + D_{n-j}^{(i)}W_{2,j}^{(i)}(z)\right] \tag{3.13}$$

where $A_{n-j}^{(i)}$, $B_{n-j}^{(i)}$, $C_{n-j}^{(i)}$, $D_{n-j}^{(i)}$, $1 \le i \le M$ – unknown functions which can be found from boundary conditions and $G_{1,j}^{(i)}(z)$, $W_{1,j}^{(i)}(z)$, $G_{2,j}^{(i)}(z)$, $W_{2,j}^{(i)}(z)$ – respectively systems of fundamental solutions of sequences (3.3) and (3.4). Using the unknown coefficients method they can be represented in the form:

$$G_{l,j}^{(i)}(z) = \exp(-\varsigma_{l,i}z)\sum_{k=0}^{j}a_{j,k}^{l,i}\frac{(\omega_{l,i}z)^k}{k!}\ ;$$

$$W_{l,j}^{(i)}(z) = \exp(\varsigma_{l,i}z)\sum_{k=0}^{j}a_{j,k}^{l,i}\frac{(-\omega_{l,i}z)^k}{k!}\ , \tag{3.14}$$

where coefficients $a_{j,k}^{l,i}$ can be determined from simple recurrent expressions:

$$a_{j,k+1}^{l,i} = \frac{\omega_{l,i}}{2\varsigma_{l,i}}\left[a_{j,k+2}^{l,i} - \sum_{p=k}^{j-1}(j-p+1)a_{p,k}^{l,i}\right] \tag{3.15}$$

under arbitrary $a_{j,0}^{l,i}$. In the expressions (3.15) $j = 1,2,\ldots,k = 0,1,\ldots,j-k$ and all $a_{j,k}^{l,i}$ for $j < k$ are identically equal to zero.

Since functions $W_{l,i}^{(i)}(z)$ unlimitedly increase while $z\to\infty$ it's evident that according to conditions (3.11) it should be taken:

$$B_j^{(M)} \equiv 0,\ \ D_j^{(M)} \equiv 0,\ \ j=0,1,2,\ldots \tag{3.16}$$

We use conditions (3.5)-(3.10) for the determination of the unknown functions $A_{n-j}^{(i)}$, $B_{n-j}^{(i)}$, $C_{n-j}^{(i)}$, $D_{n-j}^{(i)}$. For this we substitute representation (3.12), (3.13) into the condition (3.5). Using the meaning for $\partial_z^2\theta_n^{(1)}$ from the equation (3.3) and after regrouping we obtain:

$$\sum_{j=0}^{n}\left[A_j^{(1)}\left\{\frac{\xi^2}{\tilde{\eta}_1^2}G_{1,n-j}^{(1)}(0) + \frac{\lambda^2\eta_1^2}{2}\sum_{m=j}^{n}G_{1,m-j}^{(1)}(0)\right\} + B_j^{(1)}\left\{\frac{\xi^2}{\tilde{\eta}_1^2}W_{1,n-j}^{(1)}(0) + \frac{\lambda^2\eta_1^2}{2}\sum_{m=j}^{n}W_{1,m-j}^{(1)}(0)\right\} +\right.$$
$$\left. + C_j^{(1)}d_zG_{2,n-j}^{(1)}(0) + D_j^{(1)}d_zW_{2,n-j}^{(1)}(0)\right] = \frac{\eta_1^2}{2}\lambda^2\sum_{m=0}^{n}(n-m+1)\bar{p}_z^m\ . \tag{3.17}$$

Let $a_{0,0}^{l,i} = 1$, $a_{j,0}^{l,i} = 0$, $j = 1, 2, ...,$ $1 \le i \le M$. In the left part of the equation (3.17) we leave only expressions for $j = n$, the rest we write in the right part. As a result we obtain:

$$A_n^{(1)}\left(\frac{\xi^2}{\tilde{\eta}_1^2}+\frac{\lambda^2\eta_1^2}{2}\right)+B_n^{(1)}\left(\frac{\xi^2}{\tilde{\eta}_1^2}+\frac{\lambda^2\eta_1^2}{2}\right)-C_n^{(1)}\varsigma_{2,1}+D_n^{(1)}\varsigma_{2,1}=\frac{\lambda^2\eta_1^2}{2}\sum_{m=0}^{n}(n-m+1)\bar{p}_z^m - $$
$$-\sum_{j=0}^{n-1}\left[\left(A_j^{(1)}+B_j^{(1)}\right)\frac{\lambda^2\eta_1^2}{2}(n-j+1)+C_j^{(1)}d_z G_{2,n-j}^{(1)}(0)+D_j^{(1)}d_z W_{2,n-j}^{(1)}(0)\right]. \quad (3.18)$$

Using the same technique as for equations (3.6) – (3.10) we receive the recurrent sequence of the equation systems of the dimension $(4M-2)\times(4M-2)$. With the help of these systems the unknown functions $A_{n-j}^{(i)}$, $B_{n-j}^{(i)}$, $C_{n-j}^{(i)}$, $D_{n-j}^{(i)}$ can be determined. Thus, the construction of the solution of the problem (2.5)-(2.14) is completed.

The final expressions for functions $\theta^{(i)}(x,z,\tau)$, $\varphi^{(i)}(x,z,\tau)$ and nonzero components of displacements vector received from Fourier and Laguerre integral transformations inversion formulae:

$$\theta^{(i)}(x,z,\tau)=\int_0^\infty \bar{\theta}^{(i)}(\xi,z,\tau)\cos(x\xi)d\xi \,;$$

$$\varphi^{(i)}(x,z,\tau)=\int_0^\infty \bar{\varphi}^{(i)}(\xi,z,\tau)\cos(x\xi)d\xi \,;$$

$$w^{(i)}(x,z,\tau)=\int_0^\infty \bar{w}^{(i)}(\xi,z,\tau)\cos(x\xi)d\xi \,;$$

$$u^{(i)}(x,z,\tau)=\int_0^\infty \frac{\bar{\theta}^{(i)}(\xi,z,\tau)-\partial_z\bar{w}^{(i)}(\xi,z,\tau)}{\xi}\sin(x\xi)d\xi \,,$$

where $\{\bar{\theta}^{(i)}, \bar{\varphi}^{(i)}, \bar{w}^{(i)}\} = \lambda\sum_{n=0}^{\infty}\{\bar{\theta}^{(i)}, \bar{\varphi}^{(i)}, \bar{w}^{(i)}\}L_n(\lambda\tau)$ and for calculation of $\bar{w}_n^{(1)}$ we can use formulae:

$$\bar{w}_n^{(i)}=\frac{1}{\tilde{\eta}_i^2\lambda^2}\left[\bar{\Phi}_n^{(i)}-2\bar{\Phi}_{n-1}^{(i)}+\bar{\Phi}_{n-2}^{(i)}+d_z\bar{\theta}_n^{(i)}-2d_z\bar{\theta}_{n-1}^{(i)}+d_z\bar{\theta}_{n-2}^{(i)}\right].$$

Using the known displacements the stress tensor components can be determined according to Hook's law:

$$\sigma_{zz}^{(i)}/\mu_i=\left(\eta_i^2-2\right)\theta^{(i)}+2\partial_z w^{(i)}\,;$$

$$\sigma_{xx}^{(i)}/\mu_i=\left(\eta_i^2-2\right)\theta^{(i)}+2\partial_x u^{(i)}\,;$$

$$\sigma_{xz}^{(i)}/\mu_i=\partial_z u^{(i)}+\partial_x w^{(i)}\,,$$

where μ_i is the shear modulus of the i-th layer.

4. NUMERICAL SOLUTION

In this example we show the results of calculation of displacement $w^{(1)}(x,z,\tau)$ and normal stresses $\sigma_{zz}^{(1)}(x,z,\tau)$ on the interfaces of elastic layer and half-space loaded by shock effect on the surface $z = 0$, when in the area $|x| \le 1$ the normal pressure is given by the law:

$$p(x,\tau)=p^*\sqrt{1-x^2}S_+(\tau)\,,$$

and in the area $|x| > 1$ equal to zero.

The calculation refers for condition with the following parameters $z_1 = \frac{h_1}{h} = 0.5$; $\tilde{\eta}_1 = 1$, $\tilde{\eta}_2 = 0.943$, $\Omega_1 = 1.543$, $\eta_1 = \eta_1 = 1.528$.

The displacements and stresses are the suitable solution to related of statically problem. It is based on the Laguerre integral transform method with limit transition. For example we may write, $w_{st}^{(1)}(x,z) = \lim_{\lambda\to 0} w_0^{(1)}(x,z)$, where $w_{st}^{(1)}(x,z)$ – is the displacements in the static problem, $w_0^{(1)}(x,z)$ is the value of the transform $w_n^{(1)}(x,z)$ when $n = 0$.

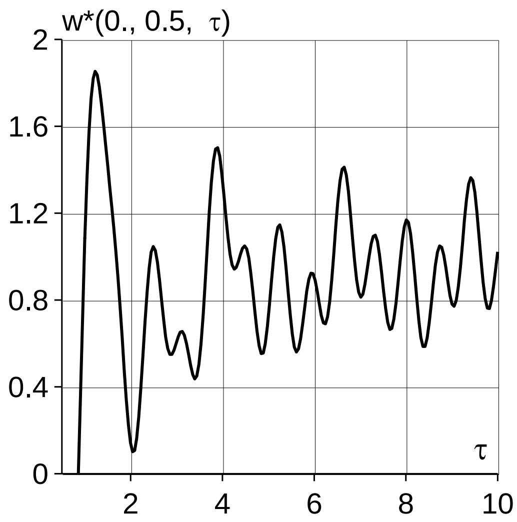

Fig. 1. Time distribution of normal displacement at the boundary of coating and half-space

Figures 1 and 2 give the dimensionless displacements $w^*(x,z,\tau) \equiv \frac{w^{(1)}(x,z,\tau)}{w_{st}^{(1)}(x,z)}$ and stresses on the interface of elastic layer and half-space at the point $x = 0$. As seen from the results of the time variation of displacements and normal stresses in the process of transition is in the form of oscillations around the static equilibrium position (unit values).

The first wave of the displacements and stresses arrives at and it is in accordance with the physics of the phenomenon. The peak value of amplitude gets its maximum and soon after arrival of the compression wave and thereafter the amplitudes decrease. The extreme values of amplitudes are attained in the beginning of transitional period when the compression wave-reflection from the interfaces of elastic layer and half-space arrives. The multiple reflection results increase of frequency-wave.

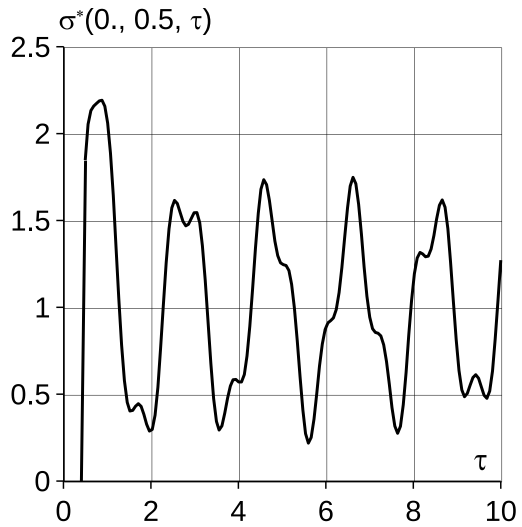

Fig. 2. Time distribution of normal stresses at the boundary of coating and half-space

The numerical results have demonstrated the advantages of the present theory in terms of effectiveness and efficiency, which seem to justify its more intensive formulation. Future work will address the extension of the present theory for the dynamic problems of layered bodies.

5. CONCLUSION

The paper proposes a new mathematical formulation of the plane dynamic problem of elasticity theory for a layered half-space. The formulation involves non-classical separation of equations of motion using two functions with a particular mechanical meaning – volumetric expansion and function of acceleration of the shift. In terms of these functions obtained two wave equation, written boundary conditions and the conditions of ideal mechanical contact of layers. Using the Laguerre and Fourier integral transformations was obtained the solution of the formulated problem. The results of the calculation of the stress-strain state in the half-space with a coating for a local impact loading are presented.

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