BENDING OF ORTHOTROPIC PLATE CONTAINING A CRACK PARALLEL TO THE MEDIAN PLANE

Vasyl SHVABYUK*, Iaroslav PASTERNAK*, Heorhiy SULYM**

* Lutsk National Technical University, Lvivska Str., 74, 43018 Lutsk, Ukraine
** Bialystok Technical University, Faculty of Mechanical Engineering, Wiejska Str., 45C, 15-351 Bialystok, Poland

pasternak@ukrpost.ua, h.sulym@pb.edu.pl

Abstract: This paper considers cylindrical bending of the plate containing a crack parallel to plate's faces. The analytical model of the problem is obtained using the improved theory of plates bending, which considers transverse deformation of the plate. Received analytical results are compared with the numerical data of the boundary element approach, which is modified to suit the considered contact problem. The results of analytical and numerical techniques are in a good agreement both for the isotropic and anisotropic plates.

1. INTRODUCTION

The problem of analysis of thin plates weakened by cracks is especially important in the case of composite materials, due to the possibility of interlayer delaminating. However, crack growth parallel to the median surface of plate is less dangerous than the perpendicular crack growth, the problem of analysis of such element is still actual. This problem is studied in the monographs by Panasyuk et al., (1975), Marchuk and Homyak (2003), Serensen and Zaytsev (1982), Cherepanov (1983), etc. Some of the problems for edge cracks are solved by numerical methods. One can see them in the well-known handbook edited by Murakami (1987). In the study of Gnuni and Yegnazarian (2002) stability and bending problems of thin plates containing internal cracks are examined under the classical bending theory conditions. In the present work, the problem of cylindrical bending of plate with internal crack is solved basing on the equations of the improved theory of the middle thickness plate bending (Shvabyuk, 1974). The influence of transversal anisotropy and length of the crack on stress and displacement of the plate is studied.

2. STATEMENT OF THE PROBLEM. BOUNDARY CONDITIONS

Cylindrical bending of the plate of a thickness $2h$ is considered. The plate is hinge-supported on the edges $x = \mp a$. The plate is weakened by a symmetric tunnel internal crack (at $-l \le x \le l$), which is placed at the depth of $z = h - h_0$ parallel to the median surface (Fig. 1).

The plate is bended with the uniform load q, which is applied at the outer surface $z = -h$. To solve the stated problem one can utilize the technique (Gnuni and Yegnazarian, 2002), according to which the plate is formally decomposed into two domains with different bending rigidities: the domain containing a crack, which cylindrical rigidity equals the algebraic sum of rigidities of the upper and lower plate elements:

$$D_1 = D_1^- + D_1^+ = \delta D$$
$$(\delta = 1 - 3\beta + 3\beta^2,\ \beta = h_0/2h); \tag{1}$$

and a domain without a crack, which cylindrical rigidity (1) equals $D_2 = D = 2Eh^3/3(1-\nu^2)$. Thus, $D_1^+ = \tilde{E}(2h - h_0)^3/12 = (1-\beta)^3 D$ is a rigidity of the upper plate part over the crack; and $D_1^- = \beta^3 D$ is a rigidity of the lower plate part under the crack; $\tilde{E} = E/(1-\nu)^2$; E is an elasticity modulus; ν is a Poisson ratio.

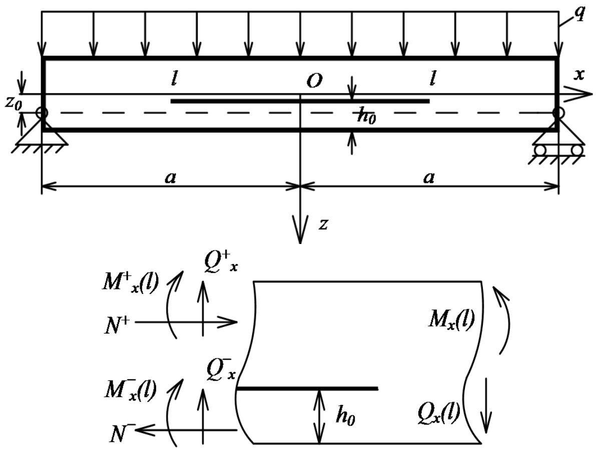

Fig. 1. Scheme of plate loading

It should be noted that the used technique can be applied in cases, when the plate model does not take into account the transverse compression, i.e. when vertical displacements do not depend on the transverse coordinate z. Within this technique it is impossible to determine the real normal stress σ_x, which act in the upper and lower parts of the plate over and under the crack, respectively. Therefore, henceforward (with the use of equations (1), (2)) a model of plates of medium thickness (Shvabyuk, 1974), which utilize the improved equations of bending, is used. Corresponding equations for the vertical displacement $W(x,z)$ and normal stress σ_x allow studying the stress-strain state more precisely and satisfy the boundary conditions for each part of the plate both on the domains' interface and on the faces of the plate.

Assume that the contact pressure p, which acts on the crack faces is constant along the whole crack and such that it can be obtained using the displacement of the middle surface w_l of the lower part of the plate under the crack by classic formula for deflection under cylindrical bending: $p = D_1^- w_l^{\mathrm{IV}}$. For the upper part of the plate and for the whole cracked domain this equation can be written through corresponding displacements w_u and w in the following form: $q - p = D_1^+ w_u^{\mathrm{IV}}$ and $q = (D_1^+ + D_1^-) w^{\mathrm{IV}} = \delta D w^{\mathrm{IV}}$. Neglecting on this stage of the transversal compression ($w_l = w_u = w$) one can obtain an approximate equation for contact pressure p on the crack faces:

$$p = \frac{D_1^- q}{D_1^+ + D_1^-} = \frac{\beta^3 q}{1 - 3\beta + 3\beta^2}. \tag{2}$$

Taking into account that $\sigma_z = -p$ and substituting $\beta = 1/2(1 - z/h)$ in the equation (2) one can obtain expression for contact stress σ_z as a function of transversal coordinate z:

$$\sigma_z = \frac{q}{2} \frac{(z/h - 1)^3}{(1 + 3z^2/h^2)}. \tag{3}$$

For estimation of stress-strain state of the plate, equations of improved model of transtropic (transversally-isotropic) plates (Shvabyuk, 1974), which take into account both the transverse shear and transverse compression, are used:

$$\begin{aligned} &D_i w_i^{\mathrm{IV}} = q_{i2} - \varepsilon_1 h_i^2 q_{i2}^{\mathrm{II}} - \varepsilon_2 h_i^4 q_{i2}^{\mathrm{IV}}, \\ &K_i' w_{i\tau}^{\mathrm{II}} = -q_{i2}, \quad E u_i^{\mathrm{II}} = -\nu''(1+\nu) q_{i1}^{\mathrm{I}}, \end{aligned} \tag{4}$$

where: $D_i = D = I\tilde{E}$, $I = 2h^3/3$, $K' = 4G'h/3$, $q_{i2} = q^-$, $q_{i1} = -0{,}5q^-$, $u_i, w_i, w_{i\tau}, h_i = u, w, w_\tau, h$ for the domain $|x| > l$; u_i, $w_i, w_{i\tau} = u_u$, $w_u, w_{u\tau}$; $q_{i1} = q_{u1} = (\sigma_z(h - h_0) - q^-)/2 = -(p + q^-)/2$; $q_{i2} = q_{u2} = q^- + \sigma_z(h - h_0) \equiv q^- - p = q(1 - (\beta^3/(1 - 3\beta + 3\beta^2)))$; $\beta = h_0/2h$; $D_i = D_1^+ = I_1^+ \tilde{E}$; $I_1^+ = 2h(1-\beta)^3/3$; $h_1 = h_1^+ = (1 - \beta)$ or $D_i = D_1^- = I_1^- \tilde{E}$, $I_1^+ = h_0^3/12$; $K'_l = 2G'h_0/3$; $q_{i1} = q_{l1} = -0{,}5p$; $q_{i2} = q_{l2} = q\beta^3/(1 - 3\beta + 3\beta^2)$; u_i, $w_i, w_{i\tau}, h_i = u_l$, $w_l, w_{l\tau}, h_0/2$ for the domain $|x| \le l$; $\varepsilon_2 = 1/20(1-\alpha)\tilde{E}/E'$, $\tilde{E} = E/(1-\nu^2)$; $\alpha = 0{,}5\nu'' G'/G$; E, E', G', ν'' are elastic moduli and Poisson ratio of plate material in the longitudinal and transverse (with primes) directions; $q^- = q$ is a distributed load applied to the upper surface of the plate ($z = -h$); u is a horizontal displacement of the median surface of the plate; w, w_τ are total and shear components of vertical displacement of plat median surface; Roman numeral superscripts of w, w_τ, u and q_1, q_2 denote the order of derivative by the variable x; subscripts "u" and "l" denote respectively upper and lower parts of the plate at the cracked domain; $2h$ is a height of cross-section of the plate; h_0 is a thickness of the plate part which is under the crack. Further, the case of $q^+ = 0$ is considered.

Expressions for stresses σ_x, σ_z and displacements $U(x,z)$, $W(x,z)$ of the plate outside the cracked domain, according to this model are as follows (Shvabyuk, 1974):

$$\begin{aligned} &\sigma_x = \frac{N_x}{2h} + \frac{M_x}{I} z \\ &+ \frac{z(z^2 - 0.6h^2)}{3I(1-\nu)} \left(\frac{G}{G'} - \nu''\right)\left(q_2 - 0.5 q_2'' h^2 \frac{G'}{E'}\right); \\ &\sigma_z = q_1 + \frac{1}{4}\left(3\frac{z}{h} - \frac{z^3}{h^3}\right) \cdot q_2; \\ &q_1 = \frac{1}{2}(q^+ - q^-), \quad q_2 = (q^+ + q^-); \\ &U(x,z) = u(x) \\ &- z\left(\frac{dw}{dx} - \frac{dw_\tau}{dx}\left(1 - (1-\alpha)\frac{z^2}{3h^2}\right)\right) + \frac{(1-\alpha)}{8E'h}\frac{dq_2}{dx} z^3; \\ &W(x,z) = w(x) \\ &+ 2\alpha_0 z \cdot \frac{q_1}{E'} + A' \cdot \frac{d^2 w}{dx^2} \cdot \frac{z^2}{2} + \frac{\alpha_0 \cdot q_2}{8E'h} \cdot B(z), \end{aligned} \tag{5}$$

where: $B(z) = 6A_2 z^2 - A_3 \dfrac{z^4}{h^2}$; $A' = \dfrac{\nu''}{(1-\nu)}$;

$\alpha_0 = 0.5 - \nu' \cdot A'$, $\tilde{w} = w + 1.5\varepsilon_2 q_2 h/\tilde{E}$, $A_2 = 1 + \dfrac{A'E'}{2\alpha_0 G'}$;

$A_3 = A_2 - \dfrac{\nu'' A' E'}{4\alpha_0 G}$; $M_x = \displaystyle\int_{-h}^{h} z\sigma_x dz = -D\frac{d^2\tilde{w}}{dx^2} - \varepsilon_1 h^2 q_2$,

$Q_x = K' \dfrac{dw_\tau}{dx}$, $N_x = \displaystyle\int_{-h}^{h} \sigma_x dz = 2\tilde{E}h\frac{du}{dx} + 2A'hq_1$

are a bending moment, transverse and longitudinal forces in the plate; u is a tangential displacement of median surface of the plate.

The system of equations (4) is solved separately for each domain of the plate. Herewith, the corresponding boundary conditions are satisfied joining the solutions for each section, and the conditions of problem symmetry are taken into account.

In particular, for the domain $|x| > l$ the following relations hold:

$$\begin{aligned} &w = C_0 + C_2 x^2 + qx^4/(24D); \\ &w_\tau = C_\tau - qx^2/(2K'); \; u = \nu''(1+\nu)qx/(2E) + R_0 \end{aligned} \tag{6}$$

For the domain $|x| \le l$, if it concerns the lower part of the plate under the crack, which face is loaded with the normal stress $\sigma_Z(h-h_0) = -q\beta^3/(1-3\beta+3\beta^2)$, the displacement (3) are as follows:

$$w_l = C_{l0} + C_{l2}x^2 + q_{l2}x^4/\left(24D_1^-\right);$$
$$w_{l\tau} = C_{l\tau} - q_{l2}x^2/\left(2K_l'\right);\ u_l = R_1 x + R_{l0} \tag{7}$$

Here constants $C_0, C_2, C_\tau, R_0, C_{l0}, C_{l2}, C_{l\tau}, R_1, R_{l0}$ are obtained using the boundary conditions on the edge $x = a$ of the plate:

$$w(a) = M_x(a) = N_x(a) = 0;\ Q_x(a) = -qa \tag{8}$$

and joining the solutions on the cracked domain boundary $x = l$ for upper and lower parts of the plate, which are studied as separate objects, loaded (except external loading) with additional surface contact pressure p in the cracked domain. For example, for the lower part of the plate:

$$w_l(l) = W(l, h-h_0/2);\ u_l(l) = U(l, h-h_0/2);$$
$$\sigma_x^-(l, h_0/2) = \sigma_x(l,h);\ N_x^-(l) = \int_{h-h_0}^{h}\sigma_x dz, \tag{9}$$

where:

$$\sigma_x^-\left(x, z^-\right) = \frac{N_x^-}{h_0} + \frac{M_x^-}{I^-}z^-$$
$$+\frac{z^-\left((z^-)^2 - 0.15h_0^2\right)}{3I^-(1-\nu)}\left(\frac{G}{G'} - \nu''\right)\left(q_{l2} - 0.125q_{l2}''h_0^2\frac{G'}{E'}\right);$$

$$N_x^- = h_0\tilde{E}u_l' + h_0A'q_{l1};\ M_x^- = -D_1^-\tilde{w}_l'' - 0.25\varepsilon_1 h_0^2 q_{l2};$$

$$Q_x^-(l) = Q_x(l) + q_{u2}l,\ Q_x(l) = -ql;\ z^- = z - h + h_0/2$$

is a thickness coordinate of the lower plate part under the crack.

Satisfying the boundary conditions (5), one can obtain the factors C_0, C_2, C_τ along with the equations for the bending moment M_x and displacements w and w_τ in the uncracked domain:

$$C_0 = 5qa^4/(24D) + q\varepsilon_1 a^2h^2/(2D),$$
$$C_2 = -qa^2/(4D) - q\varepsilon_1 h^2/(2D),$$
$$C_\tau = qa^2(1+2\varepsilon_1 h^2/a^2)/(2K');$$
$$M_x = q(a^2 - x^2)/2;\ Q_x(x) = -qx; \tag{10}$$
$$w = q(x^4 - a^4)/(24D)$$
$$+qa^2\left(a^2 - x^2\right)(1+2\varepsilon_1 h^2/a^2)/(4D);$$
$$w_\tau = qa^2(1+2\varepsilon_1 h^2/a^2)/(2K') - qx^2/(2K')$$

Utilizing conditions (6) one receives the rest of unknown factors:

$$C_{l0} = W(l, h-h_0/2) - C_{l2}l^2 - q_{l2}l^4/\left(24D_1^-\right);$$

$$C_{l2} = -(2M_{lx}(l) + q_{l2}l^2 + 0.5\varepsilon_1 q_{l2}h_0^2)/\left(4D_1^-\right);$$

$$R_0 = \left(R_1 - \frac{A'q}{2\tilde{E}}\right)l + \frac{3}{4}\frac{qla^2}{\tilde{E}h^2}\left(\frac{l^2}{3a^2} - 2\varepsilon_1\frac{h^2}{a^2} - 1\right) + \frac{ql}{4G'}f(\beta)$$

$$R_1 = N_{lx}(l)/\left(\tilde{E}h_0\right) - A'q_{l1}/\tilde{E};\quad C_{l\tau} = -\frac{2D^-}{K_l'}C_{l2};$$

$$R_{l0} = 0;\quad f(\beta) = (1-\beta)[3-(1-\beta)^2(1-\alpha)];$$

$$W(l, h-h_0/2) = w(l)$$
$$+3\nu''(1+\nu)(1-\beta)^2\left(\frac{l^2}{h^2} - \frac{a^2}{h^2} - 2\varepsilon_1\right)\frac{qh}{8E}$$
$$-\alpha_0(1-\beta)[8 - f_1(\beta)A_2 - \tilde{A}_2]\frac{qh}{8E'}; \tag{11}$$

$$\tilde{A}_2 = \frac{(\nu'')^2(1-\beta)^3E'}{2(1-\nu-2\nu'\nu'')G};$$

$$N_x^-(l) = 3\beta(1-\beta)M_x(l)/h - \frac{(G/G' - \nu'')qh}{4(1-\nu)}f_2(\beta);$$

$$M_x^-(l) = -N_x^-(l)h_0/6 + 0.25M_x(l)h_0^2/h^2$$
$$+0.2h_0^2(q - q_{l2})(G/G' - \nu'')/6(1-\nu);$$

$$f_1(\beta) = (1-\beta)(5+2\beta-\beta^2)];$$

$$f_2(\beta) = (1-2\beta)^2\left(2\beta^2 - 2\beta - 0.1\right) + 0.1.$$

3. BASIC EQUATIONS

Thus, proceeding from equations (6) – (8) the resultant forces and bending moment for the part of plate under the crack are as follows:

$$N_x^-(x) = N_x^-(l);\quad Q_x^-(x) = -q_{l2}x;$$
$$M_x^-(x) = M_x^-(l) + \frac{q_{l2}}{2}\left(l^2 - x^2\right) \tag{12}$$

Taking into consideration that the longitudinal forces in the plate parts above the crack and under the crack $(x \le l)$ are equal in magnitude and are opposite in sign $N_x^-(x) = -N_x^+(x)$, the value of the boundary bending moment $M_x^+(l)$ for the upper plate part $x \le l$ is obtained from the boundary condition of equality of normal stresses on the outer $(z = -h)$ surface of the plate:

$$\sigma_x^+(l, -h(1-\beta)) = \sigma_x(l, -h) \tag{13}$$

Thus,

$$M_x^+(l) = -(1-\beta)N_x^-(l)h/3 + (1-\beta)^2M_x(l)$$
$$+0.4h^2(1-\beta)^2(q - q_{u2})\left(\frac{G}{G'} - \nu''\right)\Big/3(1-\nu). \tag{14}$$

Together these quantities have to satisfy the equation of the moments' balance in the plate at the cracked domain (Fig. 1b):

$$M_x^+(l)+M_x^-(l)+dN_x^-(l)=M_x(l). \qquad (15)$$

Here d is a distance between the points of application of longitudinal forces to the transverse crosscuts of the plate parts; $d=h$ for the linear distribution of the normal stresses σ_x.

It should be noted that the equation of the moments balance (12) includes a term, which takes into account the influence of longitudinal forces and which was not taken into consideration by G.P. Cherepanov in his "general theory of delaminating of the multilayer shells" (Cherepanov, 1983, p. 267). It was explained by the fact that the distance d was considered to be less then the linear size of the crack ($d \ll l$).

At the same time an account of the longitudinal forces $N_x^-(l)$, acting in the transverse cross-section of the upper and lower plate parts, allow, under the condition of static equilibrium of plate part, which is "cut" along crack plane, and on its extension, to determine the shear (tangent) force, acting at the extension of the crack:

$$T(l)=\int_l^a \tau_{xz}^0 dx = N_x^-(l). \qquad (16)$$

where $\tau_{xz}^0=-3\beta(1-\beta)Q_x/h$ is a "background" stress acting on the horizontal cross-section of the plate without a crack but on its depth.

Approximate value of stress intensity factor (SIF) K_{II} can be determined by the formula:

$$K_{\mathrm{II}}=\tau_{xz}^0(l)\sqrt{2\pi l}=-3\sqrt{2}\beta(1-\beta)\frac{Q_x(l)}{h}\sqrt{\pi l}. \qquad (17)$$

Maximal ($x=0$) normal displacements w_l (without account of compression), as well as a stress σ_x, that appear on the external and internal surfaces of the lower plate part can be written as follows:

$$w_l=\frac{5qa^4}{24D}\left[\frac{\left(\theta^2+2,4\varepsilon_1\beta^2h^2/a^2\right)\theta^2}{1-3\beta+3\beta^2}\right.$$
$$\left.+2.4\varepsilon_1(1-\theta^2)h^2/a^2+1-\theta^4\right];$$
$$\sigma_x^-(0,h_0/2)=\frac{3}{4}\frac{a^2}{h^2}\left(1-\frac{1-4\beta+3\beta^2}{1-3\beta+3\beta^2}\theta^2\right)q$$
$$+\frac{1}{5}\frac{(G/G'-\nu'')}{(1-\nu)}q; \qquad (18)$$
$$\sigma_x^-(0,-h_0/2)=\frac{3}{4}\frac{qa^2}{h^2}[(1-2\beta)(1-\theta^2)$$
$$-\frac{\beta\theta^2}{1-3\beta+3\beta^2}]-\frac{1}{4}\frac{(G/G'-\nu'')}{(1-\nu)}q\left(\frac{1}{\beta}f_2(\beta)+\frac{4}{5}\right).$$

In the case, when the crack is placed along the median surface of the plate ($\beta=0{,}5$; $h_0=h$), edge bending moments $M_{ux}(l)$, $M_{lx}(l)$ and longitudinal force $N_{lx}(l)$, acting at the edges of plate parts in the cracked domain, are expressed through the bending moment $M_x(l)$ of the whole plate by the following formulas:

$$M_x^-(l)=M_x^+(l)=\frac{1}{8}M_x(l)+\frac{qh^2}{48}\frac{(G/G'-\nu'')}{(1-\nu)};$$
$$N_x^-(l)=\frac{3}{4}M_x(l)/h-\frac{qh}{40}\frac{(G/G'-\nu'')}{(1-\nu)} \qquad (19)$$

Consider an extreme case, when the crack is located on the median surface of a plate. Formulas for maximal normal stresses (in the cross-section $x=0$) on the internal and outer surfaces of the plate parts, divided by the crack, take the following form:

$$\sigma_x(0,\pm h)=\pm\frac{3}{4}\frac{a^2}{h^2}\left(1+\theta^2\right)q\pm\frac{1}{5}\frac{(G/G'-\nu'')}{(1-\nu)}q$$
$$\sigma_x(0,\mp 0)=\mp\frac{3}{2}\frac{a^2}{h^2}\theta^2 q\mp\frac{1}{4}\frac{(G/G'-\nu'')}{(1-\nu)}q \qquad (20)$$

where $\theta=l/a$ is relative length of the crack; upper and lower signs of notations "$\pm$" and "$\mp$" in formulas (17) correspond to the outer surfaces of the lower and upper parts of the plate, respectively. Stress $\sigma_x(0,\mp 0)$ acts on the internal surfaces of the plate parts located under (sign "–") and above (sign "+") the crack.

Maximal displacement of the median surface of the lower part of the plate can be written in the following form:

$$w_l=\frac{5qa^4}{24D}\left[1+3\theta^4-0.3A'(1-\theta^2)\frac{h^2}{a^2}\right.$$
$$\left.+2.4\varepsilon_1\left(1-0.25A'h^2/a^2\right)\frac{h^2}{a^2}\right]-\frac{\alpha_0 qh\tilde{B}}{16E'}, \qquad (21)$$

where $\alpha_0=\frac{1}{2}-\frac{\nu'\nu''}{1-\nu}$, $\tilde{A}=5.125-\frac{\nu''(46E'/G'+\nu''E'/G)}{16(1-\nu-2\nu'\nu'')}$.

In expression (18) terms with multipliers h^2/a^2 and α_0 are corrections to the classical thin plate theory, that take into account transverse shear and compression. Assuming that they are zero ones, one can obtain the simplest approximate expression for calculation of the vertical displacement of a thin plate containing a crack at its median surface:

$$w_l=\frac{5qa^4}{24D}\left(1+3\theta^4\right). \qquad (22)$$

If the cylindrical stiffness D in the latter equation is replaced with the value EI, one obtains the formula for a vertical displacement of the beam of a constant section containing a crack along the middle line. However, this expression is not precise enough for thick plates and short beams, especially those made of composite materials. In this case it is necessary to use the complete formula (18) along with the correspondent expression (2) for displacement W, utilizing certain corrections for parameters q_1, q_2 and h.

This problem can be also solved using a hypothesis based on one of the Timoshenko-type theories (taking into account the expression for σ_z). However, in this case formulas (15) – (17) don't contain second term, which accounts the influence of transverse shear and compression on the nonlinearity of distribution of the normal stress σ_x. Correction, which accounts the transverse shear and compression, in formulas (15) – (17) is a constant value, though it is not constant for stresses on the external and internal surfaces of the parts of the plate and it also depends on the plate material.

4. NUMERICAL MODELING OF THE PROBLEM. DUAL BOUNDARY ELEMENT METHOD

Numerical modeling of the considered problem is used for verification of the obtained results. Dual boundary element method (Portela et al., 1992) for the plane elastostatics is utilized for this purpose. It is well-known that classical boundary element method degenerate when considering crack problems due to the lack of equations considering load of crack faces (Portela et al., 1992). Therefore, the dual boundary element method (Portela et al., 1992) was developed, which proceed with a system of $2n$-equations basing on Somigliana identity: n displacement equations (as in classical BEM) and additionally n stress equations obtained from the Somigliana identity by differentiation. Thus, for the problems of cracks theory, dual BEM integral equations take the following form (Portela et al., 1992):

– for collocation point "**y**" placed on a smooth surface Γ of a solid –

$$\begin{aligned}\frac{1}{2}u_i(\mathbf{y}) &= \int_\Gamma U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\int_\Gamma T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{x})d\Gamma(\mathbf{x})\\ &+\int_{\Gamma_C^+} U_{ij}(\mathbf{x},\mathbf{y})\Sigma t_j(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\int_{\Gamma_C^+} T_{ij}(\mathbf{x},\mathbf{y})\Delta u_j(\mathbf{x})d\Gamma(\mathbf{x});\end{aligned} \tag{23}$$

– for collocation point "**y**" placed on a smooth surface Γ_C^+ of a crack:

$$\begin{aligned}\frac{1}{2}\Sigma u_i(\mathbf{y}) &= \int_\Gamma U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\int_\Gamma T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{x})d\Gamma(\mathbf{x})\\ &+\int_{\Gamma_C^+} U_{ij}(\mathbf{x},\mathbf{y})\Sigma t_j(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\int_{\Gamma_C^+} T_{ij}(\mathbf{x},\mathbf{y})\Delta u_k(\mathbf{x})d\Gamma(\mathbf{x});\\ \frac{1}{2}\Delta t_i(\mathbf{y}) &= \int_\Gamma D_{ijk}(\mathbf{x},\mathbf{y})n_j^+(\mathbf{y})t_k(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\int_\Gamma S_{ijk}(\mathbf{x},\mathbf{y})n_j^+(\mathbf{y})u_k(\mathbf{x})d\Gamma(\mathbf{x})\\ &+\int_{\Gamma_C^+} D_{ijk}(\mathbf{x},\mathbf{y})n_j^+(\mathbf{y})\Sigma t_k(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\int_{\Gamma_C^+} S_{ijk}(\mathbf{x},\mathbf{y})n_j^+(\mathbf{y})\Delta u_k(\mathbf{x})d\Gamma(\mathbf{x}).\end{aligned} \tag{24}$$

Here **x** is an arbitrary point of the surface; U_{ij}, T_{ij}, D_{ijk}, S_{ijk} are the singular and hypersingular kernels of integral equations for plane problem of elasticity, which are explicitly written in Portela et al. (1992), u_i, t_i are the components of displacement and traction vectors; $\Delta u_i = u_i^+ - u_i^-$, $\Delta t_i = t_i^+ - t_i^-$, $\sum u_i = u_i^+ + u_i^-$, $\sum t_i = t_i^+ + t_i^-$; n_j^+ are the components of a unit normal vector to a surface Γ_C^+; signs "+" and "-" denote the values concerned with the surfaces Γ_C^+ and Γ_C^-, formed by a cut Γ_C. Subscripts in notations correspond to the projections of vectors on the axis of global coordinate system Ox_1x_2. Einstein summation convention is assumed. Kernels of integral equations for the plane problem elasticity at $\mathbf{x}\to\mathbf{y}$ possess the following singularities:

$$\begin{aligned}&U_{ij}(\mathrm{x},\mathbf{y})\sim \ln|\mathbf{x}-\mathbf{y}|, \quad T_{ij}(\mathrm{x},\mathrm{y})\sim 1/\ln|\mathbf{x}-\mathbf{y}|\\ &D_{ijk}(\mathrm{x},\mathbf{y})\sim 1/\ln|\mathbf{x}-\mathbf{y}|, \quad S_{ijk}(\mathbf{x},\mathbf{y})\sim (1/\ln|\mathbf{x}-\mathbf{y}|)^2\end{aligned} \tag{25}$$

For modeling of closed cracks, the equation (20) should be modified with account of additional conditions of zero value of normal displacement discontinuities and shear contact stresses on the mathematical cut Γ_C as follows:

$$\begin{aligned}&\Delta u_n(\mathbf{y}) = \Omega_{1j}(\mathbf{y})\Delta u_j(\mathbf{y})\big|_{\mathbf{y}\in\Gamma_C} \equiv 0\\ &\tau = \Omega_{2j}(\mathbf{y})\Delta t_j(\mathbf{y})\big|_{\mathbf{y}\in\Gamma_C} \equiv 0 \quad \Sigma t_j(\mathbf{y})\big|_{\mathbf{y}\in\Gamma_C} \equiv 0\end{aligned} \tag{26}$$

Here components of rotation tensor Ω of the vectors equal:

$$\begin{aligned}&\Omega_{11}(\mathbf{y}) = n_1(\mathbf{y}), \quad \Omega_{12}(\mathbf{y}) = n_2(\mathbf{y}),\\ &\Omega_{21}(\mathbf{y}) = -n_2(\mathbf{y}), \; \Omega_{22}(\mathbf{y}) = n_1(\mathbf{y})\end{aligned} \tag{27}$$

Thus, equation (21) on a mathematical cut Γ_C should be solved for the unknown discontinuities $\Delta u_\tau = \Omega_{2j}(\mathbf{y})\Delta u_j(\mathbf{y})$ of tangent displacement and normal contact stress $p_n = -1/2\Omega_{1j}(\mathbf{y})\Delta t_j(\mathbf{y})$. Proceeding from this, the following system of integral equations can be obtained:

$$\begin{aligned}&\frac{1}{2}\Sigma u_\tau(\mathbf{y}) = \Omega_{2i}(\mathbf{y})\Big[\int_\Gamma U_{ij}t_j d\Gamma - \int_\Gamma T_{ij}u_j d\Gamma\\ &-\int_{\Gamma_C^+} T_{ij}(\mathbf{x},\mathbf{y})\Omega_{2j}(\mathbf{x})\Delta u_\tau(\mathbf{x})d\Gamma(\mathbf{x})\Big],\\ &p_n(\mathbf{y}) = -n_i^+(\mathbf{y})n_j^+(\mathbf{y})\Big[\int_\Gamma D_{ijk}t_k d\Gamma(\mathbf{x}) + \int_\Gamma S_{ijk}u_k d\Gamma\\ &+\int_{\Gamma_C^+} S_{ijk}(\mathbf{x},\mathbf{y})\Omega_{2j}(\mathbf{x})\Delta u_\tau(\mathbf{x})d\Gamma(\mathbf{x})\Big],\\ &0 = \Omega_{2i}(\mathbf{y})n_j^+(\mathbf{y})\Big[\int_\Gamma D_{ijk}t_k d\Gamma(\mathbf{x}) + \int_\Gamma S_{ijk}u_k d\Gamma\\ &+\int_{\Gamma_C^+} S_{ijk}(\mathbf{x},\mathbf{y})\Omega_{2j}(\mathbf{x})\Delta u_\tau(\mathbf{x})d\Gamma(\mathbf{x})\Big].\end{aligned} \tag{28}$$

The following numerical solution procedure for integral equations (20), (25) using the dual BEM is proposed. For evaluation of curvilinear integrals, curves Γ, Γ_C are divided into parts, which are approximated with the rectilinear sections Γ_q (boundary elements). Thus, equations (20), (25) are written as sums of integrals along boundary elements Γ_q. n nodal points $\mathbf{x}^{q,p}(p=\overline{1,n})$ are set on each element Γ_q. As a rule, discontinuous boundary elements (Portela et al., 1992), i.e. elements with no node placed at the end point of a boundary element, are used to solve crack theory problems.

Particularly, often used are rectilinear quadratic boundary elements with three nodes placed as follows: one in the center and the other two at the distance of $1/3$ of element length from the central point. This allows modeling of the non-smooth surfaces, because collocation point never coincide the corner or a brunching point of a cut Γ_C. Boundary functions t_i, u_i, p_n, Δu_τ are interpolated on element Γ_q using their node values as follows:

$$\left[\tilde{t}_i, \tilde{u}_i, \tilde{p}_n, \Delta\tilde{u}_\tau\right](\xi) = \sum_{p=1}^{n}\left[t_i^{q,p}, u_i^{q,p}, p_n^{q,p}, \Delta u_\tau^{q,p}\right]\phi^p(\xi) \quad (29)$$

$$\tilde{u}_i(\xi) = \sum_{p=1}^{n} u_i^{q,p}\phi^p(\xi) \quad (30)$$

where ξ is a parameter of a point position at the boundary element, defined on the interval $-1 \le \xi \le 1$: $d\Gamma_q = (L_q/2)d\xi = J_q d\xi$, J_q is a Jacobian of a variable change on Γ_q. For a rectilinear quadratic discontinuous boundary element ($n = 3$ the values of the parameter $\xi = \{-2/3; 0; 2/3\}$ correspond to its nodes $\mathbf{x}^{q,p}$. Thus, interpolation polynomials $\phi^p(\xi)$ are expressed as follows:

$$\phi^1 = \xi\left(\frac{9}{8}\xi - \frac{3}{4}\right) \; \phi^2 = \left(1 - \frac{3}{2}\xi\right)\left(1 + \frac{3}{2}\xi\right)$$
$$\phi^3 = \xi\left(\frac{9}{8}\xi + \frac{3}{4}\right) \quad (31)$$

Thus, the system of singular integral equations is reduced to a system of linear algebraic equations, which is sought for the nodal values of boundary functions. Obtained solutions of the integral equation system are used for calculation of stress σ_x/q, displacement $\widetilde{w} = w_l E/(2qh)$ and stress intensity factor (SIF) $K_{\mathrm{II}}^* = K_{\mathrm{II}}/\sqrt{\pi l}$.

5. NUMERICAL RESULTS

Analysis of numerical results of plane problem of elasticity and formula (17) for stresses σ_x allows to state that the growth of crack in a plate causes the increase of stresses in parts of the plate according to formulas (17) in compliance with quadratic parabola law, while plane elasticity behaves according to the rule close to hyperbola law. One can see that in the second case the growth is much slower. Therefore, formulas (17) should be modified by replacing parameter θ^2 with parameter θ^4 and written in the following form:

$$\sigma_x(0, \pm h) = \pm\frac{3}{4}\frac{a^2}{h^2}\left(1 + \theta^4\right)q \pm \frac{1}{5}\frac{\left(G/G' - \nu''\right)}{(1-\nu)}q$$
$$\sigma_x(0, \mp 0) = \mp\frac{3}{2}\frac{a^2}{h^2}\theta^4 q \mp \frac{1}{4}\frac{\left(G/G' - \nu''\right)}{(1-\nu)}q \quad (32)$$

To prove that such modification is reasonable, the values of stress σ_x^-/q on the outer surface of the lower part of the plate is evaluated utilizing formula (17) and formula (28), placed in a separate column in bold font (for $\theta = 0$ and $\theta = 1$ they are the same), and using dual BEM for plane problem of elasticity (in brackets). These results are presented in Tables 1 and 2.

Tab. 1. Stress values in isotropic plate

$\frac{a}{h}$	σ_x^- / q (*isotropy*)			
	$\theta = 0$	$\theta = 0.5$		$\theta = 1$
5	18.95 (18.85)	23.64 (20.58)	***20.12***	37.7 (38.6)
10	75.20 (75.00)	93.95 (81.69)	***79.89***	150.2 (152.6)
20	300.2 (299.7)	375.2 (325.7)	***308.9***	600.2 (610.4)

Tab. 2. Stress values in transtropic plate

$\frac{a}{h}$	σ_x^- / q (*wood*)		
	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
5	21.52 (21.03)	***22.69*** (22.69)	40.27 (38.30)
10	77.77 (76.51)	***82.46*** (83.35)	152.8 (148.6)
20	302.8 (301.7)	***321.52*** (328.7)	602.8 (596.3)

The value of contact pressure on crack faces, which is determined by stress σ_z completely coincides with corresponding numerical results of the plane problem of elasticity along the whole length of the cracks except a small area near the crack tips.

Fig. 2 shows plots of stress σ_x/q versus the transverse coordinate $z/2h$ for parameters $\theta = 0{,}5$ and $\theta = 0{,}9$, at $a/h = 10$, obtained using the improved formulas (28) (solid line), and the dual boundary element method of plane problem of elasticity (dashed line). Dash-dot line presents the corresponding plot for plate without a crack. Data, obtained by dual BEM of plane problem of elasticity, are presented in Fig. 2 in brackets.

Data analysis for stresses, presented in Tab. 1 and in Fig. 2 for isotropic and transtropic (wood) materials, prove that formulae of applied theory of medium thickness plates are quite precise (in comparison with numerical data of plane elasticity, the error is less than 2.5%) and convenient for calculations.

Analysis of formulas (15) – (19) shows that growth of crack length increases stresses and displacements in the plate up to the values which can appear in two separate plates put one onto another without friction. Then the stresses in them will increase twice and displacements in four times. Corrections, which account transverse shear and compression, are insignificant for stresses in case of isotropic material. At the same time these corrections may be important for transtropic materials (fiberglass plastics, wood, etc.). For example, for wood ($G/G' = 10; \upsilon'' = \upsilon = 0{,}3$), when $a/h = 0{,}5$; $\theta = 0{,}5$, errors of classical theory for the first and second formulas are as big as 12% and 37%, respectively. These errors are even bigger, when determining the maximal vertical displacement $w \cong w_l$. See comparative table for relative displacements $\widetilde{w} = w_l E/(2gh)$ below (Table 2).

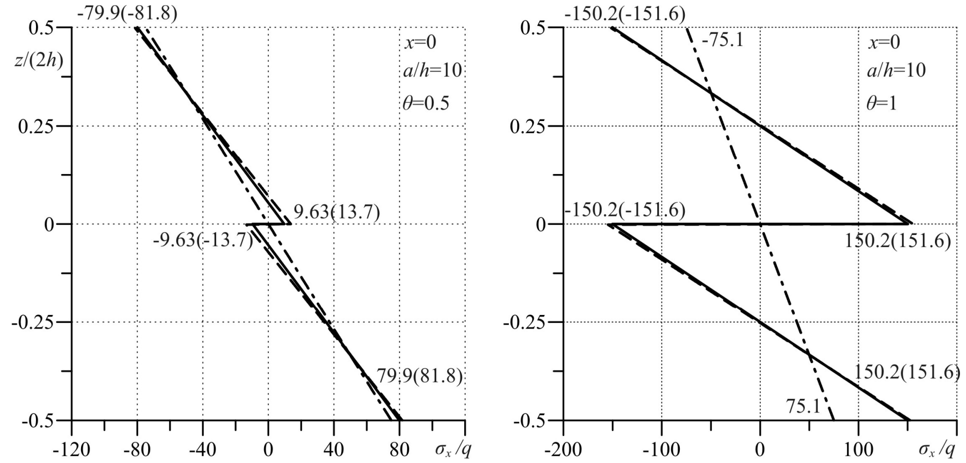

Fig. 2. Plots of stress σ_x/q against the transverse coordinate $z/2h$

Tab. 3. Values of vertical displacements for isotropic plate

$\frac{a}{h}$	$\tilde{w} = w_l E/(2qh)$ (isotropy)					
	$\theta = 0$ (*p.p.*)	$\theta = 0$	$\theta = 0.5$ (*p.p.*)	$\theta = 0.5$	$\theta = 1$ (*p.p.*)	$\theta = 1$
5	100.8	96.85(88.87)	105.4	114.3(105.5)	384.9	365.5
10	1459	1454(1422)	1522	1723(1688)	5917	5728
20	$22.86\cdot 10^3$	$22.88\cdot 10^3$	$23.79\cdot 10^3$	$27.15\cdot 10^3$	$94.13\cdot 10^3$	$91.16\cdot 10^3$

Tab. 4. Values of vertical displacements for wooden board

$\frac{a}{h}$	$\tilde{w} = w_l E/(2qh)$ (wood)						
	$\theta = 0$ (*p.p.*)	$\theta = 0$	$\theta = 0.5$ (*p.p.*)	$\theta = 0.5$	$\theta = 1$ (*p.p.*)	$\theta = 1$	$\theta = 1$ (*cl.*)
5	184.9	184.3	192.6	201.5	446.7	452.9	355.5
10	1755	1805	1840	2073	5910	6078	5688
20	$24.01\cdot 10^3$	$24.28\cdot 10^3$	$25.10\cdot 10^3$	$28.56\cdot 10^3$	$90.92\cdot 10^3$	$92.56\cdot 10^3$	$91.0\cdot 10^3$

Data in Tables 3 and 4 in brackets present the displacements calculated according to the simplified formula (16) for thicknesses $a/h = 5; 10$ without account of transverse shear and compression. These data are shown in the last column of Table 3 and are the same both for transtropic (wood) and for isotropic materials. Values, calculated using the dual boundary element method of plane problem of elasticity, are put into columns (p.p.). The laws of maximum result deviation of displacements $\tilde{w}$, calculated using the applied theories of plates, are similar to those detected while stress calculations. Thus, the real character of displacements growth in a plate caused by crack length increase are, in fact, much slower than it is determined by the formulas of applied theories of plates. This is of special importance for the following values of parameter θ: $0{,}5 \leq \theta \leq 0$.

6. CONCLUSION

This paper obtains the analytic dependences convenient for engineering applications and calculation of stresses and displacements in isotropic and transversally-isotropic plates, damaged by horizontal cracks. These results allow predicting with enough practical accuracy the strength and rigidity of plates using the geometrical parameters of a crack in a plate, as well as physical characteristics of material and its transversal anisotropy.

REFERENCES

1. **Cherepanov G.P.** (1983), *Fracture mechanics of composite materials*, Nauka, Moscow (in Russian).
2. **Gnuni V.C., Yegnazarian T.A.** (2002), On strength of plates weakened by cracks, *In collected articles «Optimal control over durability and strength of mechanical systems»*. Yerevan. Publishing house of Yerevan State University, 37-41 (in Russian).
3. **Marchuk M.V., Homyak M.M.** (2003), *On a mixed finite element analysis calculation of composite laminated shells and plates*, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics NAS of Ukraine, L'viv. (in Ukrainian).
4. **Murakami Y. ed.** (1987) *Stress intensity factors handbook*, Vols. 1, 2. – Pergamon Press, New York.
5. **Panasyuk V.V., Savruk M.P., Datsyshyn A.P.** (1975), *Distribution of strains in crack areas in plates and membranes*, Naukova Dumka, Kyiv (in Russian).
6. **Portela A., Aliabadi M.H., Rooke D.P.** (1992) The dual boundary element method: Effective implementation for crack problems, *Int. J. Numer. Meth. Engng*., 33, P. 1269–1287.
7. **Serensen S.V., Zaytsev G.P.** (1982), *Bearing capacity of thin-slab structures built of reinforced plastics with defects*, Naukova Dumka, Kyiv (in Russian).
8. **Shvabyuk V.I.** (1974), On one variant of generalized theory of transversally-isotropic plates, *Prikladnaya Mekhanika*, V. 10, No. 11, P.87-92 (in Russian).