REDUCTION AND DECOMPOSITION OF SINGULAR FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

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Abstract: Reduction of singular fractional systems to standard fractional systems and decomposition of singular fractional discrete-time linear systems into dynamic and static parts are addressed. It is shown that if the pencil of singular fractional linear discrete-time system is regular then the singular system can be reduced to standard one and it can be decomposed into dynamic and static parts The proposed procedures are based on modified version of the shuffle algorithm and illustrated by numerical examples.

1. INTRODUCTION

Singular (descriptor) linear systems have been addressed in many papers and books (Dodig and Stosic, 2009; Dai, 1989; Fahmy and O'Reill, 1989; Gantmacher, 1960; Kaczorek, 1992, 2007a; Kucera and Zagalak, 1988). The eigenvalues and invariants assignment by state and output feedbacks have been investigated in (Dodig and Stosic, 2009; Dai, 1989; Fahmy and O'Reill, 1989; Kucera and Zagalak, 1988; Kaczorek, 2004) and the realization problem for singular positive continuous-time systems with delays in Kaczorek (2007b). The computation of Kronecker's canonical form of a singular pencil has been analyzed in Van Dooren (1979). The fractional differential equations have been considered in the monograph (Podlubny, 1999). Fractional positive linear systems have been addressed in (Kaczorek, 2008, 2010) and in the monograph (Kaczorek, 2011). Luenberger in (Luenberger, 1978) has proposed the shuffle algorithm to analysis of the singular linear systems.

In this paper a modified version of the shuffle algorithm will be proposed for the reduction of the singular fractional system to equivalent standard fractional system and for decomposition of the singular fractional system into dynamic and static parts.

The paper is organized as follows. In section 2 it is shown that if the pencil of the singular system is regular then the singular system can be reduced to equivalent standard fractional system. The decomposition of singular fractional system into dynamic and static parts is addressed in section 4. Concluding remarks are given in section 5.

To the best of the author's knowledge the reduction and the decomposition of singular fractional linear discrete-time systems have not been considered yet.

The following notation be used in the paper.

The set of $n\times m$ real matrices will denoted by $\Re^{n\times m}$ and $\Re^n := \Re^{n\times 1}$. The set of $m\times n$ real matrices with nonnegative entries will be denoted by $\Re_+^{mxn}$ and $\Re_+^n := \Re_+^{nx1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix by I_n.

2. REDUCTION OF SINGULAR FRACTIONAL SYSTEMS TO EQUIVALENT STANDARD FRACTIONAL SYSTEMS

Consider the singular fractional discrete-time linear system described by the state equation:

$$E\Delta^{\alpha} x_{i+1} = Ax_i + Bu_i\,,\ i\in Z_+ = \{0,1,...\} \tag{1}$$

where, $x_i\in\Re^n, u_i\in\Re^m$ are the state and input vectors, $A\in\Re^{n\times n}, E\in\Re^{n\times n}, B\in\Re^{n\times m}$ and the fractional difference of the order α is defined by:

$$\Delta^{\alpha} x_i = \sum_{k=0}^{i}(-1)^k\binom{\alpha}{k}x_{i-k}\,,\ 0<\alpha<1 \tag{2}$$

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for}\quad k=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for}\quad k=1,2,... \end{cases} \tag{3}$$

It is assumed that:

$$\det E = 0 \tag{4a}$$

and

$$\det[Ez - A] \neq 0 \tag{4b}$$

for some $z\in C$ (the field of complex numbers).

Substituting (2) into (1) we obtain:

$$\sum_{k=0}^{i+1} Ec_k x_{i-k+1} = Ax_i + Bu_i\,,\ i\in Z_+ \tag{5}$$

where:

$$c_k = (-1)^k\binom{\alpha}{k} \tag{6}$$

Applying the row elementary operations to (5) we obtain:

$$\sum_{k=0}^{i+1}\begin{bmatrix}E_1\\0\end{bmatrix}c_k x_{i-k+1} = \begin{bmatrix}A_1\\A_2\end{bmatrix}x_i + \begin{bmatrix}B_1\\B_2\end{bmatrix}u_i\,,\ i\in Z_+ \tag{7}$$

where $E_1 \in \Re^{n_1 \times n}$ is full row rank and $A_1 \in \Re^{n_1 \times n}$, $A_2 \in \Re^{(n-n_1) \times n}$, $B_1 \in \Re^{n_1 \times m}$, $B_2 \in \Re^{(n-n_1) \times m}$. The equation (7) can be rewritten as:

$$\sum_{k=0}^{i+1} E_1 c_k x_{i-k+1} = A_1 x_i + B_1 u_i \tag{8a}$$

and

$$0 = A_2 x_i + B_2 u_i \tag{8b}$$

Substituting in (8b) i by $i + 1$ we obtain:

$$A_2 x_{i+1} = -B_2 u_{i+1} \tag{9}$$

The equations (8a) and (9) can be written in the form:

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_1 - c_1 E_1 \\ 0 \end{bmatrix} x_i - \begin{bmatrix} c_2 E_1 \\ 0 \end{bmatrix} x_{i-1} - \dots - \begin{bmatrix} c_{i+1} E_1 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} u_{i+1} \tag{10}$$

If the matrix:

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \tag{11}$$

is nonsingular then premultiplying the equation (10) by the inverse matrix $\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1}$ we obtain the standard system:

$$x_{i+1} = \bar{A}_0 x_i + \bar{A}_1 x_{i-1} + \dots + \bar{A}_i x_0 + \bar{B}_0 u_i + \bar{B}_1 u_{i+1} \tag{12}$$

where:

$$\bar{A}_0 = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 - c_1 E_1 \\ 0 \end{bmatrix}, \quad \bar{A}_1 = -\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} c_2 E_1 \\ 0 \end{bmatrix}, \dots, \quad \bar{A}_i = -\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} c_{i+1} E_1 \\ 0 \end{bmatrix},$$
$$\bar{B}_0 = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -B_2 \end{bmatrix}. \tag{13}$$

If the matrix (11) is singular then applying the row elementary operations to (10) we obtain:

$$\begin{bmatrix} E_2 \\ 0 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{20} \\ \bar{A}_{20} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ \bar{A}_{21} \end{bmatrix} x_{i-1} + \dots + \begin{bmatrix} A_{2,i} \\ \bar{A}_{2,i} \end{bmatrix} x_0 + \begin{bmatrix} B_{20} \\ \bar{B}_{20} \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ \bar{B}_{21} \end{bmatrix} u_{i+1} \tag{14}$$

where $E_2 \in \Re^{n_2 \times n}$ is full row rank with $n_2 \geq n_1$ and $A_{2,j} \in \Re^{n_2 \times n}$, $\bar{A}_{2,j} \in \Re^{(n-n_2) \times n}$, $j = 0,1,\dots,i$, $B_{2,k} \in \Re^{n_2 \times m}$, $\bar{A}_{2,k} \in \Re^{(n-n_2) \times m}$, $k = 0,1$.

From (14) we have:

$$0 = \bar{A}_{20} x_i + \bar{A}_{21} x_{i-1} + \dots + \bar{A}_{2,i} x_0 + \bar{B}_{20} u_i + \bar{B}_{21} u_{i+1} \tag{15}$$

Substituting in (15) i by $i + 1$ (in state vector x and in input u) we obtain:

$$\bar{A}_{20} x_{i+1} = -\bar{A}_{21} x_i - \dots - \bar{A}_{2,i} x_1 - \bar{B}_{20} u_{i+1} - \bar{B}_{21} u_{i+2} \tag{16}$$

From (2.14) and (2.16) we have:

$$\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{20} \\ -\bar{A}_{21} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ -\bar{A}_{22} \end{bmatrix} x_{i-1} + \dots + \begin{bmatrix} A_{2,i} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_{20} \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ -\bar{B}_{20} \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ -\bar{B}_{21} \end{bmatrix} u_{i+2} \tag{17}$$

If the matrix:

$$\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix} \tag{18}$$

is nonsingular then premultiplying the equation (17) by its inverse we obtain the standard system:

$$x_{i+1} = \hat{A}_0 x_i + \hat{A}_1 x_{i-1} + \dots + \hat{A}_i x_0 + \hat{B}_0 u_i + \hat{B}_1 u_{i+1} + \hat{B}_2 u_{i+2} \tag{19}$$

where:

$$\hat{A}_0 = \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} A_{20} \\ -\bar{A}_{21} \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} A_{21} \\ -\bar{A}_{22} \end{bmatrix}, \dots, \quad \hat{A}_i = \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} A_{2,i} \\ 0 \end{bmatrix},$$
$$\hat{B}_0 = \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} B_{20} \\ 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} B_{21} \\ -\bar{B}_{20} \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\bar{B}_{21} \end{bmatrix} \tag{20}$$

If the matrix (18) is singular we repeat the procedure. Continuing this procedure after at most n steps we finally obtain a nonsingular matrix and the desired fractional system. The procedure can be justified as follows. The elementary row operations do not change the rank of the matrix $[Ez - A]$. The substitution in the equations (8b) and (15) i by $i + 1$ also does not change the rank of the matrix $[Ez - A]$ since it is equivalent to multiplication of tits lower rows by z and by assumption (4b) holds. Therefore, the following theorem has been proved.

Theorem 1. The singular fractional linear system (5) satisfying the assumption (4) can be reduced to the standard fractional linear system

$$x_{i+1} = \tilde{A}_0 x_i + \tilde{A}_1 x_{i-1} + \dots + \tilde{A}_i x_0 + \tilde{B}_0 u_i + \tilde{B}_1 u_{i+1} + \dots + \tilde{B}_p u_{i+p} \tag{21}$$

where $\tilde{A}_j \in \Re^{n \times n}$, $j = 0,1,\dots,i$, $\tilde{B}_k \in \Re^{n \times m}$, $k = 0,1,\dots,p < n$ whose dynamics depends on the future inputs $u_{i+1},\dots,u_{i+p}$.

Example 1. Consider the singular fractional linear system (1) for $\alpha = 0,5$ with:

$$E = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2 & 2 & -2 \\ 2 & 1 & 0 \\ -1.8 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}. \tag{22}$$

In this case the conditions (4) are satisfied since: $\det E = 0$ and

$$\det[Ez-A]=\begin{vmatrix} 5z-0.2 & -2 & 2z+2 \\ 2z-2 & -1 & z \\ z+1.8 & 0 & 1 \end{vmatrix}=z-0.2$$

Applying to the matrices (22) the following elementary row operations $L[1+2\times(-2)]$, $L[3+1\times(-1)]$ we obtain:

$$[E \quad A \quad B]=\begin{bmatrix} 5 & 0 & 2 & 0.2 & 2 & -2 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 1 & 0 & 0 & -1.8 & 0 & -1 & 2 & -1 \end{bmatrix}$$

$$\to\begin{bmatrix} 1 & 0 & 0 & -3.8 & 0 & -2 & 3 & -2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 & -1 & 1 \end{bmatrix} \tag{23}$$

$$=\begin{bmatrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix}$$

and the equations (8) have the form:

$$\sum_{k=0}^{i+1} c_k \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} x_{i-k+1}=\begin{bmatrix} -3.8 & 0 & -2 \\ 2 & 1 & 0 \end{bmatrix} x_i+\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} u_i \tag{24a}$$

and

$$0=[2 \quad 0 \quad 1]x_i+[-1 \quad 1]u_i \tag{24b}$$

Using (6) we obtain:

$$c_1=-\binom{\alpha}{1}=-\alpha=-0.5,\ c_2=(-1)^2\binom{\alpha}{2}=\frac{\alpha(\alpha-1)}{2!}=-\frac{1}{8},$$

$$\dots,\ c_{i+1}=(-1)^{i-1}\left.\frac{\alpha(\alpha-1)\dots(\alpha-i)}{(i+1)!}\right|_{\alpha=0.5}$$

and the equation (10) has the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} x_{i+1}=\begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x_i+\frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_{i-1}$$
$$-\dots-c_{i+1}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_0+\begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} u_i+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1} \tag{25}$$

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ is singular and we perform the elementary row operation $L[3+2\times(-1)]$ on (25) obtaining the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_{i+1}=\begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ -3 & -1 & -0.5 \end{bmatrix} x_i+\frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} x_{i-1}$$
$$-\dots-c_{i+1}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} x_0+\begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 1 & -2 \end{bmatrix} u_i+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1} \tag{26}$$

The matrix:

$$\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix} \tag{27}$$

is nonsingular and we obtain the equation (19) with the matrices:

$$\hat{A}_0=\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1}\begin{bmatrix} A_{20} \\ -\bar{A}_{21} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1}\begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ 0.25 & 0 & 0.125 \end{bmatrix}$$
$$=\begin{bmatrix} -3.3 & 0 & -2 \\ 4.85 & -0.5 & 3.625 \\ 9.6 & 1 & 4.5 \end{bmatrix}$$

$\vdots$

$$\hat{A}_i=\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1}\begin{bmatrix} -A_{2,i} \\ 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$=\begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}, \tag{28}$$

$$\hat{B}_0=\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1}\begin{bmatrix} B_{20} \\ 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1}\begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}$$
$$=\begin{bmatrix} 3 & -2 \\ -5.5 & 3 \\ -7 & 6 \end{bmatrix}$$

$$\hat{B}_1=\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1}\begin{bmatrix} B_{21} \\ -\bar{B}_{20} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}$$
$$=\begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\hat{B}_2=\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -\bar{B}_{21} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$$
$$=\begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

3. DECOMPOSITION OF SINGULAR FRACTIONAL SYSTEM INTO DYNAMIC AND STATIC PARTS

Consider the singular fractional system (5) satisfying the assumptions (4). Applying the procedure presented in section 2 after p steps we obtain:

$$\begin{bmatrix} E_p \\ 0 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{p,0} \\ \bar{A}_{p,0} \end{bmatrix} x_i + \begin{bmatrix} A_{p,1} \\ \bar{A}_{p,2} \end{bmatrix} x_{i-1} + \ldots + \begin{bmatrix} A_{pi} \\ \bar{A}_{pi} \end{bmatrix} x_0 + \begin{bmatrix} B_{p,0} \\ \bar{B}_{p,0} \end{bmatrix} u_i + \begin{bmatrix} B_{p,1} \\ \bar{B}_{p,1} \end{bmatrix} u_{i+1} + \ldots + \begin{bmatrix} B_{p,p-1} \\ \bar{B}_{p,p-1} \end{bmatrix} u_{i+p-1} \quad (29)$$

where $E_p \in \Re^{n_p \times n}$ is full row rank, $A_{pj} \in \Re^{n_p \times n}$, $\bar{A}_{pj} \in \Re^{(n-n_p)\times n}$, $j = 0,1,\ldots,p$ and $B_{pk} \in \Re^{n_p \times m}$, $\bar{B}_{pk} \in \Re^{(n-n_p)\times m}$, $k = 0,1,\ldots,p-1$ with nonsingular matrix:

$$\begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix} \in \Re^{n \times n} \quad (30)$$

Using the elementary column operations we may reduce the matrix (30) to the form:

$$\begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}, \quad A_{21} \in \Re^{(n-n_p)\times n_p} \quad (31)$$

and performing the same elementary operations on the matrix I_n we can find the matrix $Q \in \Re^{n\times n}$ such that:

$$\begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix} Q = \begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix} \quad (32)$$

Taking into account (32) and defining the new sate vector:

$$\tilde{x}_i = Q^{-1} x_i = \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix}, \quad \tilde{x}_i^{(1)} \in \Re^{n_p}, \quad \tilde{x}_i^{(2)} \in \Re^{n-n_p}, \quad i \in Z_+ \quad (33)$$

from (29) we obtain:

$$\begin{aligned} \tilde{x}_{i+1}^{(1)} &= E_p x_{i+1} = E_p Q Q^{-1} x_{i+1} = A_{p,0} Q Q^{-1} x_i + A_{p,1} Q Q^{-1} x_{i-1} \\ &\quad + \ldots + A_{pi} Q Q^{-1} x_0 + B_{p,0} u_i + B_{p,1} u_{i+1} + \ldots + B_{p,p-1} u_{i+p-1} \\ &= [A_{p,0}^{(1)} \quad A_{p,0}^{(2)}] \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix} + [A_{p,1}^{(1)} \quad A_{p,1}^{(2)}] \begin{bmatrix} \tilde{x}_{i-1}^{(1)} \\ \tilde{x}_{i-1}^{(2)} \end{bmatrix} \\ &\quad + \ldots + [A_{pi}^{(1)} \quad A_{pi}^{(2)}] \begin{bmatrix} \tilde{x}_0^{(1)} \\ \tilde{x}_0^{(2)} \end{bmatrix} + B_{p,0} u_i + B_{p,1} u_{i+1} \\ &\quad + \ldots + B_{p,p-1} u_{i+p-1} \\ &= A_{p,0}^{(1)} \tilde{x}_i^{(1)} + A_{p,0}^{(2)} \tilde{x}_i^{(2)} + \ldots + A_{pi}^{(1)} \tilde{x}_0^{(1)} + A_{pi}^{(2)} \tilde{x}_0^{(2)} \\ &\quad + B_{p,0} u_i + B_{p,1} u_{i+1} + \ldots + B_{p,p-1} u_{i+p-1}, \quad i \in Z_+ \end{aligned} \quad (34)$$

and

$$\begin{aligned} \tilde{x}_i^{(2)} &= -A_{21} \tilde{x}_i^{(1)} - \bar{A}_{p,1}^{(1)} \tilde{x}_{i-1}^{(1)} - \bar{A}_{p,1}^{(2)} \tilde{x}_{i-1}^{(2)} - \ldots - \bar{A}_{pi}^{(1)} \tilde{x}_0^{(1)} - \bar{A}_{pi}^{(2)} \tilde{x}_0^{(2)} \\ &\quad - \bar{B}_{p,0} u_i - \ldots - \bar{B}_{p,p-1} u_{i+p-1}, \quad i \in Z_+ \end{aligned} \quad (35)$$

where:

$$A_{pj} Q = [A_{pj}^{(1)} \quad A_{pj}^{(2)}], \quad \bar{A}_{pj} = [\bar{A}_{pj}^{(1)} \quad \bar{A}_{pj}^{(2)}], \quad j = 0,1,\ldots,i \quad (36)$$

Substitution of (34) into (35) yields:

$$\begin{aligned} \tilde{x}_{i+1}^{(1)} &= \tilde{A}_{p,0} \tilde{x}_i^{(1)} + \ldots + \tilde{A}_{pi} \tilde{x}_0^{(1)} + \tilde{B}_{p,0} u_i \\ &\quad + \ldots + \tilde{B}_{p,p-1} u_{i+p-1}, \quad i \in Z_+ \end{aligned} \quad (37)$$

where

$$\begin{aligned} \tilde{A}_{p,0} &= A_{p,0}^{(1)} - A_{p,0}^{(2)} A_{21}, \ldots, \tilde{A}_{pi} = A_{pi}^{(1)} - A_{p,0}^{(2)} A_{pi}^{(1)} \\ \tilde{B}_{p,0} &= B_{p,0} - A_{p,0}^{(2)} \bar{B}_{p,0}, \ldots, \tilde{B}_{p,p-1} = B_{p,p-1} - A_{p,0}^{(2)} \bar{B}_{p,p-1} \end{aligned} \quad (38)$$

The standard system described by the equation (37) is called the dynamic part of the system (5) and the system described by the equation (35) is called the static part of the system (5).

Therefore, the following theorem has been proved.

Theorem 2. The singular fractional linear system (5) satisfying the assumption (4) can be decomposed into the dynamical part (37) and static part (35) whose dynamics depend on the future inputs $u_{i+1}, \ldots, u_{i+p-1}$.

Example 2. Consider the singular fractional system (1) for $\alpha = 0{,}5$ with the matrices (22). The matrix (27) is nonsingular. To reduce this matrix to the form (31) we perform the elementary operations $R[1+3\times(-2)]$, $R[2\times(-1)]$, $R[2,3]$. The matrix Q has the form:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -0.5 & 1 \end{bmatrix},$$

$A_{21} = [-2 \quad -0.5]$, $n_2 = 2$.

The new state vector (33) is:

$$\tilde{x}_i = Q^{-1} x_i = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} = \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix}, \quad \tilde{x}_i^{(1)} = \begin{bmatrix} x_{1,i} \\ 2x_{1,i} + x_{3,i} \end{bmatrix}, \quad \tilde{x}_i^{(2)} = -x_{2,i}. \quad (39)$$

In this case the equations (34) and (35) have the forms:

$$\begin{aligned} \tilde{x}_{i+1}^{(1)} &= \begin{bmatrix} 0.7 & -2 \\ 2 & 0.5 \end{bmatrix} \tilde{x}_i^{(1)} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tilde{x}_i^{(2)} + \frac{1}{8} \tilde{x}_{i-1}^{(1)} \\ &\quad - \ldots - c_{i+1} \tilde{x}_0^{(1)} + \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} u_i \end{aligned} \quad (40)$$

and

$$\begin{aligned} \tilde{x}_i^{(2)} &= [2 \quad 0.5] \tilde{x}_i^{(1)} + [0.25 \quad 0] \tilde{x}_{i-1}^{(1)} \\ &\quad + \ldots + c_{i+1} [-2 \quad 0] \tilde{x}_0^{(1)} - [1 \quad -2] u_i - [1 \quad -1] u_{i+1} \end{aligned} \quad (41)$$

Substituting (41) into (40) we obtain:

$$\tilde{x}_{i+1}^{(1)} = \begin{bmatrix} 0.7 & -2 \\ 0 & 0 \end{bmatrix} \tilde{x}_i^{(1)} + \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tilde{x}_{i-1}^{(1)}$$
$$- \ldots - c_{i+1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \tilde{x}_0^{(1)} + \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1} \tag{42}$$

The dynamic part of the system is described by (42) and the static part by (41).

4. CONCLUDING REMARKS

The singular fractional linear discrete-time systems with regular pencil have been addressed. It has been shown that if the assumption (4) are satisfied then: 1) the singular fractional linear system can be reduced to equivalent standard fractional system (Theorem 1), 2) the singular fractional linear system can be decomposed into dynamic and static parts (Theorem 2). The proposed procedures have been illustrated by numerical examples. The considerations can be easily extended to singular fractional linear continuous-time systems.

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