BEM APPROACH FOR THE ANTIPLANE SHEAR OF ANISOTROPIC SOLIDS CONTAINING THIN INHOMOGENEITIES

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Abstract: This paper considers a development of the boundary element approach for studying of the antiplane shear of elastic anisotropic solids containing cracks and thin inclusions. For modeling of thin defects the coupling principle for continua of different dimension is utilized, and the problem is decomposed onto two separate problems. The first is an external one, which considers solid containing lines of displacement and stress discontinuities and is solved using boundary element approach. The second is internal one, which considers deformation of a thin inhomogeneity under the applied load. Compatible solution of external and internal problems gives the solution of the target one. Stroh formalism is utilized to account the anisotropy of a solid and inclusion. Numerical example shows the efficiency and advantages of the proposed approach.

1. INTRODUCTION

Thin inclusions are often provided into materials for improving of their properties. Those are fibers and stringers in composite materials, glue connections, cavities and inclusions introduced for damping etc. However, thin inhomogeneities of material structure can also cause an undesirable effect. In particular, cracks, thin voids, foreign layers, thin inclusions etc. induce huge stress concentration, which can cause a failure or even mechanical fracture of the corresponding structural element.

The study of thin inhomogeneities in materials and structures mainly concerns defects of a type of thin void (crack). Application of different boundary element and boundary integral equation approaches for studying of cracks in anisotropic solids can be found in works by Ang et al. (1999), Denda and Marante (2004), Pan and Amadei (1996), Pan (1997), Sollero and Aliabadi (1995), Ting (1996) etc.

Less publications concern study of solids with thin inclusions. In modeling of thin inclusions, their influence on the main material is often replaced by the forces distributed with a certain density along a line, which lays at the median surface of inclusion (a mass forces method). Such approach is used in the BEM by Padron et al. (2007) for modeling of piles in a ground (beams bending model), by Riederer et al. (2009) for studying of anchor bolts screwed up in the rock (beam tension model), by Aliabadi and Saleh (2002) and Saleh and Aliabadi (1998) for modeling of rectilinear reinforcement of concrete. The most complete among mentioned is a model of Aliabadi and Saleh (2002) as it considers tension, shear and bending of thin inclusion. Nevertheless, the mass forces method is not suitable for modeling of the transverse deformation of inclusion, which is accompanied with displacement discontinuities at a median surface of thin inhomogeneity. Therefore, mentioned models of thin inhomogeneity require modification, which will take into account the transverse rigidity of inclusion's material.

Another approach in numerical modeling of thin inclusions is the analysis of solid with the inhomogeneity of real geometrical features and elastic properties (see Sulym and Pasternak (2008a), Sulym and Pasternak (2009)). However, in this case the thinness of inclusion should be addressed and special techniques developed by Sulym and Pasternak (2008b) are to be utilized.

The present paper develops the numerical-analytical approach for studying of the antiplane shear of anisotropic elastic solids based on the boundary element method and coupling principle for continuums of different dimension introduced by Sulym (2007).

2. PROBLEM FORMULATION

Consider a cylindrical elastic anisotropic solid, which contains a thin ribbon-like foreign inclusion. Assume that mechanical fields, which act in a solid and inclusion, and the applied load, do not depend on time and do not vary along the direction parallel to the generatrix of a solid. Consider that at the inclusion-solid interface the conditions of ideal mechanical contact are satisfied and a solid has a material symmetry plane perpendicular to the generatrix. These assumptions allow reducing spatial problem to consideration of 2D steady-state fields of a solid and inclusion acting at arbitrary plane, which is perpendicular to the solid's generatrix. The applied load is assumed to be parallel to the generatrix.

Based on the coupling principle for different dimension continua of Sulym (2007) (Fig. 1) the lined model of thin inhomogeneity can be developed. Due to the thinness of inhomogeneity its real geometrical features are withdrawn, and contact tractions and displacements are transferred onto the inclusion's median surface Γ_C (accordingly onto its faces Γ_C^+ and Γ_C^-, see Fig. 1). Thus, the problem is reduced to determination of an elastic state of a solid containing the line of discontinuities of stress and displacement fields. After development of the interaction conditions for a thin inhomogeneity along with the integral equations concerning abovementioned field discontinuities for a solid, the elastic state of the latter can be determined.

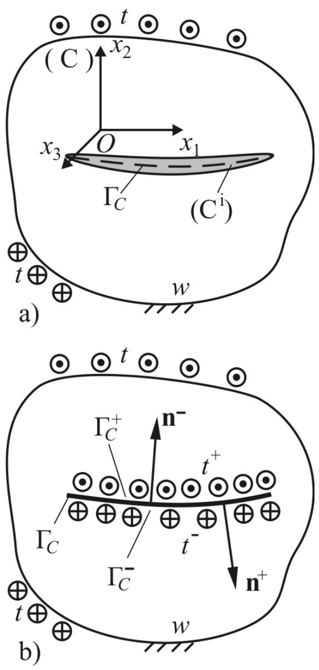

Fig. 1. Problem scheme (a) and modeling technique based on the coupling principle (b)

Consider a 2D domain S occupied by solid's median surface, which is perpendicular to the generatrix. Assume that a rectangular coordinate system $Ox_1x_2x_3$ origin is placed at this plane and Ox_3 axis is directed along the generatrix. Then, based on Ting (1996) the constitutive relations for antiplane shear of anisotropic solids can be written in the following compact form:

$$\sigma_{3j} = C_{jk} w_{,k} \quad \left(j,k = 1,2 \right), \tag{1}$$

where σ_{3j} are the only nonzero components of stress tensor; $w \equiv u_3$ is the only nonzero component of displacement vector; $C_{11} = c_{55}$, $C_{12} = C_{21} = c_{45}$, $C_{22} = c_{44}$; c_{ij} are elastic moduli of the material. Here and further the Einstein summation convention is assumed.

The equilibrium equation (see Ting (1996)) for the case of antiplane shear can be reduced to:

$$\sigma_{3j,j} + f_3 = 0, \tag{2}$$

where f_3 is a body force applied to a solid. Substituting Eq. (1) into Eq. (2) one can obtain:

$$C_{jk} w_{,jk} + f_3 = 0. \tag{3}$$

The homogeneous solution $f_3 \equiv 0$ of Eq. (3) can be obtained using the Stroh formalism. Consider that:

$$w = aF\left(x_1 + px_2\right), \tag{4}$$

where a and p are complex constants to be determined, and $F(z)$ is an analytical function of z. Substituting Eq. (4) into Eq. (3) one obtains the following eigenrelation:

$$\left[Q + 2Rp + Tp^2 \right] a = 0, \tag{5}$$

where $Q = C_{11}$, $T = C_{22}$, $R = C_{12} = C_{21}$ are numbers similar to the same-denoted Stroh matrices.

After differentiation of Eq. (4) and utilizing constitutive relations (1), stresses can be obtained in the form of Ting (1996):

$$\sigma_{31} = -\varphi_{,2}, \quad \sigma_{32} = \varphi_{,1}, \tag{6}$$

where $\varphi = bF(x_1 + px_2)$ is a stress function and

$$b = \left(R + pT\right)a = -\left(Q + pR\right)a/p. \tag{7}$$

Thus, for the solution of the problem it's more convenient to use the complex numbers a, b and p instead of elastic moduli C_{jk}. These complex numbers can be determined from the following eigenrelation of Ting (1996):

$$\mathbf{N} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1 \end{bmatrix}, \quad \begin{matrix} \mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \\ \mathbf{N}^{\mathrm{T}}\boldsymbol{\eta} = p\boldsymbol{\eta}, \end{matrix} \tag{8}$$

where $N_1 = -R/T$, $N_2 = 1/T$, $N_3 = R^2/T - Q$; $\xi = [a, b]^T$ is a right eigenvector; $\eta = [b, a]^T$ is a left eigenvector of a matrix N; superscript "T" denotes matrix transpose. Vectors ξ_α and η_β obtained for the eigenvalues p_α and p_β are normalized using the relation:

$$\boldsymbol{\xi}_\alpha^{\mathrm{T}} \boldsymbol{\eta}_\beta = \delta_{\alpha\beta}. \tag{9}$$

Therefore, the problem (8) gives two complex eigenvalues $p_1 = p$ and $p_2 = \bar{p}$, and corresponding eigenvectors $\xi_1 = \bar{\xi}_2$ are also complex conjugate. The variables w and φ are real, thus the general solution of the problem according to Ting (1996) is:

$$w = 2\,\mathrm{Re}\left[aF\left(x_1 + px_2\right) \right], \quad \varphi = 2\,\mathrm{Re}\left[bF\left(x_1 + px_2\right) \right], \tag{10}$$

Based on the Stroh formalism it is easy to obtain the Green function for a line force acting at a point $\xi(\xi_1, \xi_2)$ of infinite anisotropic medium (Ting, 1996):

$$W\left(\mathbf{x},\boldsymbol{\xi}\right) = \frac{1}{\pi} \mathrm{Im}\left[a^2 \ln Z\left(\mathbf{x},\boldsymbol{\xi}\right) \right],$$

$$T\left(\mathbf{x},\boldsymbol{\xi}\right) = \frac{1}{\pi} \mathrm{Im}\left[\frac{ab\left(n_2 - n_1 p\right)}{Z\left(\mathbf{x},\boldsymbol{\xi}\right)} \right], \tag{11}$$

where $Z(x, \xi) = x_1 + px_2 - (\xi_1, +p\xi_2)$. Displacements w along with tractions $t = \sigma_{3j} n_j$ at a point $x(x_1, x_2)$ of a solid at the surface with a normal $n(n_1, n_2)$ caused by the action of a concentrated factor $f_3\delta(\xi)$ at the point $\xi(\xi_1, \xi_2)$ can be determined within the following dependences:

$$w\left(\mathbf{x}\right) = W\left(\mathbf{x},\boldsymbol{\xi}\right) f_3, \quad t\left(\mathbf{x}\right) = T\left(\mathbf{x},\boldsymbol{\xi}\right) f_3. \tag{12}$$

Here $\delta(\boldsymbol{\xi})$ is the Dirac delta-function.

3. BOUNDARY INTEGRAL EQUATIONS OF EXTERNAL PROBLEM

Due to the symmetry of the elasticity tensor C_{jk} the following Betti type relation holds:

$$\int_{\partial S}\left[w^{(2)}t^{(1)}-t^{(2)}w^{(1)}\right]d\Gamma=\iint_{S}\left[f_3^{(2)}w^{(1)}-f_3^{(1)}w^{(2)}\right]dS \tag{13}$$

for two different stress states of a solid. Choosing field (12) as one of the states of a solid, based on Eq. (13) one can obtain Somigliana type identity for antiplane shear of anisotropic solids:

$$\begin{aligned} w(\xi)=&\int_{\partial S}\left[W(\mathbf{x},\xi)t(\mathbf{x})-T(\mathbf{x},\xi)w(\mathbf{x})\right]d\Gamma(\mathbf{x})\\ &+\iint_{S}W(\mathbf{x},\xi)f_3(\mathbf{x})dS(\mathbf{x}).\end{aligned} \tag{14}$$

Based on the method of Lin'kov (1999) of fictitious boundaries introduction and their further coupling, for a solid with a mathematical cut Γ_C one can receive the following integral representation for displacements:

$$\begin{aligned} w(\xi)=&\int_{\Gamma}\left[W(\mathbf{x},\xi)t(\mathbf{x})-T(\mathbf{x},\xi)w(\mathbf{x})\right]d\Gamma(\mathbf{x})\\ &+\int_{\Gamma_C^+}\left[W(\mathbf{x},\xi)\Sigma t(\mathbf{x})-T(\mathbf{x},\xi)\Delta w(\mathbf{x})\right]d\Gamma(\mathbf{x})\\ &+\iint_{S}W(\mathbf{x},\xi)f(\mathbf{x})dS(\mathbf{x}).\end{aligned} \tag{15}$$

Here $\sum(\cdot)=(\cdot)^+ + (\cdot)^-$; $\Delta(\cdot)=(\cdot)^+ - (\cdot)^-$; $\Gamma=\partial S$ is a boundary of a domain S; $t^{\mp}=\sigma_{3j}^{\mp}n_j^{\mp}$ ($n_j^{\mp}$ are the components of normal vectors $\mathbf{n}^{\mp}$ of surfaces $\Gamma_C^{\mp}$); signs "+" and "–" denote variables concerned with faces Γ_C^+ and Γ_C^- of the mathematical cut Γ_C, respectively.

For simplification of further notations, consider that the solid is free of body forces, i.e. $f_3\equiv 0$. Thus, the last term in Eq. (15) vanishes.

Approaching the internal source point $\boldsymbol{\xi}$ to a boundary point $\mathbf{y}\in\Gamma$ and assuming that the curve Γ is smooth at the point $\mathbf{y}$, from Eq. (15) one can obtain displacement boundary integral equation:

$$\begin{aligned} \frac{1}{2}w(\mathbf{y})=&\int_{\Gamma_C^+}\left[W(\mathbf{x},\mathbf{y})\Sigma t(\mathbf{x})-T(\mathbf{x},\mathbf{y})\Delta w(\mathbf{x})\right]d\Gamma(\mathbf{x})\\ &+\mathrm{RPV}\int_{\Gamma}W(\mathbf{x},\mathbf{y})t(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\mathrm{CPV}\int_{\Gamma}T(\mathbf{x},\mathbf{y})w(\mathbf{x})d\Gamma(\mathbf{x}),\end{aligned} \tag{16}$$

where symbols "RPV" stand for the Riemann Principal Value, and "CPV" for Cauchy Principal Value of the integral. When a collocation point y lays at the smooth surface of the mathematical cut Γ_C, one can receive the following boundary integral equation:

$$\begin{aligned} \frac{1}{2}\Sigma w(\mathbf{y})=&\mathrm{RPV}\int_{\Gamma_C^+}W(\mathbf{x},\mathbf{y})\Sigma t(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\mathrm{CPV}\int_{\Gamma_C^+}T(\mathbf{x},\mathbf{y})\Delta w(\mathbf{x})d\Gamma(\mathbf{x})\\ &+\int_{\Gamma}\left[W(\mathbf{x},\mathbf{y})t(\mathbf{x})-T(\mathbf{x},\mathbf{y})w(\mathbf{x})\right]d\Gamma(\mathbf{x}).\end{aligned} \tag{17}$$

Eq. (17) can also be used when considering the crooked cuts or thin inclusions, however the requirement that the collocation point never coincides the crook point should be provided into the numerical procedure. Differentiating Eq. (17) for y_k and using constitutive relations (1) with the account of relation $n_i^+=-n_i^-$ one can obtain the stress boundary integral equation:

$$\begin{aligned} \frac{1}{2}\Delta t(\mathbf{y})=&n_j^+(\mathbf{y})\Big[\mathrm{CPV}\int_{\Gamma_C^+}D_j(\mathbf{x},\mathbf{y})\Sigma t(\mathbf{x})d\Gamma(\mathbf{x})\\ &-\mathrm{HPV}\int_{\Gamma_C^+}S_j(\mathbf{x},\mathbf{y})\Delta w(\mathbf{x})d\Gamma(\mathbf{x})\\ &+\int_{\Gamma}\left[D_j(\mathbf{x},\mathbf{y})t(\mathbf{x})-S_j(\mathbf{x},\mathbf{y})w(\mathbf{x})\right]d\Gamma(\mathbf{x})\Big],\end{aligned} \tag{18}$$

where "HPV" stands for the Hadamard Principal Value (finite part) of an integral. The kernels of integrals in equation (18) are:

$$\begin{aligned} D_j(\mathbf{x},\mathbf{y})&=C_{jk}\frac{\partial W}{\partial y_k}=-\frac{1}{\pi}\mathrm{Im}\left[\left(\delta_{2j}-\delta_{1j}p\right)\frac{ab}{Z(\mathbf{x},\mathbf{y})}\right],\\ S_j(\mathbf{x},\mathbf{y})&=C_{jk}\frac{\partial T}{\partial y_k}=\frac{1}{\pi}\mathrm{Im}\left[\left(\delta_{2j}-\delta_{1j}p\right)\frac{b^2\left(n_2-n_1p\right)}{\left[Z(\mathbf{x},\mathbf{y})\right]^2}\right].\end{aligned} \tag{19}$$

The known expressions by Ang et al. (1999) or Pan and Amadei (1996) for kernels $D_j(\mathbf{x},\mathbf{y})$ and $S_j(\mathbf{x},\mathbf{y})$ additionally to complex numbers a and b contain constants C_{jk}, which compared to (19) complicates computational algorithms.

Assume that Γ_C is a median surface of thin inclusion. Without loss in generality, the interaction conditions between a solid and inhomogeneity, which is modeled using the coupling principle for different dimension continuums, according to Eqs. (17) and (18) is convenient to choose in the form of the following functional dependences:

$$\Sigma w(\mathbf{y})=F^w(\mathbf{y},\Sigma t,\Delta w),\quad \Delta t(\mathbf{y})=F^t(\mathbf{y},\Sigma t,\Delta w). \tag{20}$$

Thus, when the collocation point y lays at a smooth boundary Γ of a solid one can use the only Eq. (16), and when the collocation point y lays at a smooth median surface Γ_C of the inhomogeneity two integral equations (17) and (18) along with the inclusion model (20) are used.

Thus, the formulated problem is reduced to determination from the system of boundary integral equations of unknown discontinuities of displacements Δw and stress:

$$\Delta\sigma_{3n}=\left(\sigma_{3j}^+-\sigma_{3j}^-\right)n_j^+=\sigma_{3j}^+n_j^++\sigma_{3j}^-n_j^-=\Sigma t$$

at the mathematical cut Γ_C and functions w or t, which are not set by the boundary conditions at the solid's boundary Γ. After all boundary functions are obtained, one can determine the field of displacements using the integral representation (15). After differentiation of Eq. (15), one can obtain stress field at arbitrary source point $\boldsymbol{\xi}$ of a solid by:

$$\begin{aligned} \sigma_{3j}(\xi)=&\int_{\Gamma}\left[D_j(\mathbf{x},\xi)t(\mathbf{x})-S_j(\mathbf{x},\xi)w(\mathbf{x})\right]d\Gamma(\mathbf{x})\\ &+\int_{\Gamma_C^+}\left[D_j(\mathbf{x},\xi)\Sigma t(\mathbf{x})-S_j(\mathbf{x},\xi)\Delta w(\mathbf{x})\right]d\Gamma(\mathbf{x}).\end{aligned} \tag{21}$$

4. INCLUSION MODEL FOR THE INTERNAL PROBLEM

Consider mechanical fields acting at the certain cross-section y of a thin inclusion. Assume all quantities in the local coordinate system $Ox'_1x'_2x_3$, which axis Ox'_1 is directed along the normal vector $\mathbf{n}^+$. With the account of $\mathbf{n}^{\mp} = -\mathbf{n}^{i\mp}$ the conditions of a perfect mechanical contact of inclusion and a solid are $w^{\mp} = w^{i\mp}$, $t^{\mp} = -t^{i\mp}$. Here the non-italic superscript "i" denotes values concerned with the inclusion.

According to Eq. (1) stress inside the inclusion within the notations (5) equal:

$$\sigma_{31} = Q^{\mathrm{i}} w_{,1} + R^{\mathrm{i}} w_{,2}, \quad \sigma_{32} = R^{\mathrm{i}} w_{,1} + T^{\mathrm{i}} w_{,2}. \tag{22}$$

By integration of Eq. (22) over the thickness of inclusion one can obtain:

$$\begin{aligned} &\int_{-h}^{h} \sigma_{31} dh = Q^{\mathrm{i}} \left[w(h) - w(-h) \right] + R^{\mathrm{i}} \int_{-h}^{h} w_{,2} dh, \\ &\int_{-h}^{h} \sigma_{32} dh = R^{\mathrm{i}} \left[w(h) - w(-h) \right] + T^{\mathrm{i}} \int_{-h}^{h} w_{,2} dh. \end{aligned} \tag{23}$$

With the account of equilibrium equation (2) and contact conditions $t^{\mp} = -t^{i\mp}$ using the coupling principle for continuums of different dimension the following relations hold:

$$\int_{-h}^{h} \sigma_{32} dh = P(\mathbf{y}), \quad P(\mathbf{y}) = -P^0 + \int_{\mathbf{y}_0}^{\mathbf{y}} \Sigma t(s)\, ds, \tag{24}$$

where s is an arc coordinate of a mathematical cut Γ_C; P^0 is force applied at the left end of the inclusion, which position vector is defined by a point y_0. According to the mean value theorem:

$$\begin{aligned} &\int_{-h}^{h} \sigma_{31} dh = 2h\sigma_{31}^{\mathrm{avr}} \\ &\quad \approx h(\mathbf{y}) \left[t^{\mathrm{i}}(h) - t^{\mathrm{i}}(-h) \right] = h(\mathbf{y}) \Delta t(\mathbf{y}), \\ &\int_{-h}^{h} w_{,2} dh = 2hw_{,2}^{\mathrm{avr}} \\ &\quad \approx h(\mathbf{y}) \left[w_{,2}(h) + w_{,2}(-h) \right] = h(\mathbf{y}) \Sigma w_{,2}(\mathbf{y}). \end{aligned} \tag{25}$$

Withdrawing the interaction of mechanical fields acting in the directions normal and tangential to the median surface of the inhomogeneity (as in the model of the Winkler elastic foundation) and using Eqs. (24), (25) along with relations (23) one can obtain:

$$\begin{aligned} &\Delta t(\mathbf{y}) = -\frac{Q^{\mathrm{i}}(\mathbf{y})}{h(\mathbf{y})} \left[\Delta w(\mathbf{y}) + \Delta w^*(\mathbf{y}) \right], \\ &\Sigma w(\mathbf{y}) = 2w^0 + \int_{\mathbf{y}_0}^{\mathbf{y}} \frac{P(\mathbf{y}) + P^*(\mathbf{y})}{T^{\mathrm{i}}(\mathbf{y}) h(\mathbf{y})} ds. \end{aligned} \tag{26}$$

Considering assumptions made in Eq. (26) similar to Sulym (2007) the system of correcting functions Δw^*, P^* is introduced (in the theory of thin elastic defects by Sulym (2007) they are also called the end face forces and displacements). For thin defects, these functions are usually set using the aprioristic formulas. The latter are constructed under the assumption that the problem has the exact solution for three main cases of inclusion's material properties (a crack, a rigid line inclusion, and a continuous solid without inhomogeneity, i.e. inclusion material is similar to that of a medium). Considering the aprioristic formulas of Sulym (2007) the correcting functions for very thin inclusions can be assumed zero ones.

Mean value of displacement w^0 at the left end face of thin inclusion is determined using the inclusion's global equilibrium equation:

$$P^n + P^0 - \int_{\Gamma_C^+} \Sigma t(\mathbf{x})\, d\Gamma(\mathbf{x}) = 0, \tag{27}$$

where P^n is a force applied at the right end of the thin inclusion.

5. BEM SOLUTION STRATEGY

System of integral equations (16)–(18) are solved simultaneously with Eqs. (26), (27) using the boundary element method proposed by Pasternak (2011). Curves Γ and Γ_C are approximated using respectively n and n_C rectilinear segments (boundary elements Γ_q). At each element, three nodal points are set: one at the centre, and two others at the distance of 1/3 of element length at both sides of central node (discontinuous three-node boundary element; if the polynomial shape functions are used it is called discontinuous quadratic boundary element, see Saleh and Aliabadi (1998)). Thus, the collocation point never coincides the corner points of the approximated boundary and the conditions assumed for the corresponding integral equations are provided. Boundary functions t, w, $\sum t$ and Δw are approximated at the element using their nodal values.

Using the abovementioned procedure, boundary integral equations (16)–(18) along with the model of the thin inclusion (26), (27) are reduced to the system of linear algebraic equations concerning the nodal values of boundary functions t, w, $\sum t$ and Δw.

Shape functions for the elements, which do not adjoin the inhomogeneity ends, are set in the form of Lagrange polynomials for the system of nodes $\xi_p = [-2/3; 0; 2/3]$ of three-node discontinuous boundary element.

To increase the accuracy of a method and for convenient determination of the generalized stress intensity factors (SIF) it is obvious to take an advantage of the special elements, which model near-tip parts of thin inclusion. According to Pasternak (2011) the following system of shape functions is introduced for displacement discontinuities:

$$\phi_p^{\Delta w} = \Phi_{p1}^{\Delta w} \sqrt{\rho} + \Phi_{p2}^{\Delta w} \rho + \Phi_{p3}^{\Delta w} \rho^{3/2} \quad (p = 1,2,3); \tag{28}$$

and for traction discontinuities:

$$\phi_p^{\Sigma t} = \Phi_{p1}^{\Sigma t} \rho^{-1/2} + \Phi_{p2}^{\Sigma t} + \Phi_{p3}^{\Sigma t} \sqrt{\rho} \quad (p = 1,2,3). \tag{29}$$

Here ρ is the normalized distance to the inclusion's tip; $\Phi_{pj}^{\Delta w}$ and $\Phi_{pj}^{\Sigma t}$ are constant matrices, which are determined the same as the factors of Lagrange polynomials.

Shape functions (28) and (29) allow direct determination of generalized SIF K_{31} and K_{32} through equations:

$$K_{31} = \lim_{s\to 0}\sqrt{\frac{\pi}{8s}}L\cdot\Delta w(s),\quad K_{32} = -\lim_{s\to 0}\sqrt{\frac{\pi s}{2}}\Sigma t(s), \qquad (30)$$

where $L = -2\sqrt{-1}b^2$ is a real number, which correspond in 2D anisotropic elasticity to the Barnett – Lothe tensor **L** (see Ting (1996)). In the case of a crack-like defect $K_{31} = K_{III}$ and $K_{32} = 0$, where K_{III} is the classical mode III stress intensity factor.

6. NUMERICAL EXAMPLE

To show the efficiency and accuracy of the proposed approach consider the problem of a thin elastic isotropic inclusion in the infinite anisotropic medium under the uniform shear at the infinity with $\sigma_{13}^{\infty} = \sigma_{23}^{\infty} = \tau$. The relative rigidity of inclusion is characterized by the ratio $k = G^i/c_{44}$, where G^i is a shear modulus of inclusion's material. The thickness of the inclusion is assumed to be 0.001 of its length ($h = 0{,}001a$). The correcting functions are assumed to be zero ones. The scheme of the problem is depicted in Fig. 2. Two cases of medium material properties are considered: isotropic (curve 1) and orthotropic with $c_{45} = 0$ and $c_{55}/c_{44} = 10$ (curve 2). The generalized SIF obtained using Eq. (30) (solid curves) are compared with the SIF obtained by the direct approach developed by Sulym and Pasternak (2008b) (dashed curves), which uses 311 quadratic boundary elements uniformly distributed at the inclusion's interface. The results are plotted in Fig. 2.

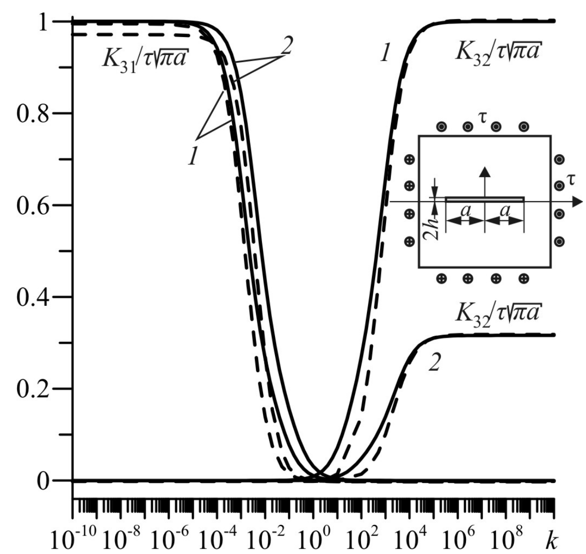

Fig. 2. Generalized SIF for a thin inclusion perfectly bonded into the isotropic (*1*) or anisotropic (*2*) medium

When the relative rigidity k of inclusion is extremely low (a cavity) or high (a rigid inclusion), the deviation of generalized SIF obtained with the proposed approach from the analytical solution of a crack or rigid line inclusion problems by Sulym (2007) is less than 0.04 % for both isotropic and anisotropic cases. The error of the direct approach is higher and about 1.5 % for isotropic and 3.5 % for anisotropic case.

For the intermediate values of the inclusion's relative rigidity the proposed approach gives a little higher values of the generalized SIF comparing to the direct one. This is explained by neglecting of correcting functions in inclusion model (26) during numerical computations. Nevertheless, higher values of generalized SIF can be treated as those, which already include safety factor.

The advantage of the proposed approach is the reduction of the general number of boundary elements (21 against 311) used for modeling of the problem. Besides, the proposed approach can be used for studying problems of very thin inclusions, which are challenging for the direct approach, even when the regularization procedure of Sulym and Pasternak (2008b) is applied.

7. CONCLUSION

Thin inhomogeneities of material structure induce considerable stress concentration at the vicinity of their tips. For thin inclusions in anisotropic elastic materials stress field near the tips of inhomogeneity possesses square root singularity. Therefore, for the numerical solution of the corresponding mathematical problem it is necessary to consider this behavior of boundary functions, in particular using special shape functions.

The stress field near the tip of thin inclusion in anisotropic elastic medium under antiplane shear can be defined within two real values, which are called generalized SIF. One of them stands for the displacement discontinuity, and in the case of a crack equal classical mode III SIF. Another stands for the stress discontinuity at the inhomogeneity.

The numerical analysis of a test problem shows high efficiency of the developed approach for modeling of different types of thin defects of anisotropic materials: cracks, thin elastic and rigid inclusions etc. It allows studying both bounded and infinite solids. The account of inclusion properties has essential influence on SIF of solids containing thin inhomogeneities.

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