CONTROL OF FRACTIONAL-ORDER NONLINEAR SYSTEMS: A REVIEW

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Abstract: This paper deals with the control of the fractional-order nonlinear systems. A list of the control strategies as well as synchronization of the chaotic systems is presented. An illustrative example of sliding mode control (SMC) of the fractional-order (commensurate and incommensurate) financial system is described and commented together with the simulation results.

1. INTRODUCTION

Control of nonlinear systems, especially *chaotic systems*, was the subject of intensive studies in the last few decades. As noted (Andrievskii and Fradkov, 2003, 2004), several thousand publications have appeared over the recent decade. It is due to the fact that chaotic behavior was discovered in numerous systems in mechanics, laser and radio physics, hydrodynamics, chemistry, biology and medicine, electronic circuits, economical systems, etc. (see (Petráš, 2011)). For this reason a natural question arises: "*How can we control chaotic systems*?"

The first important thing is that we need the mathematical formulation of chaotic processes by the basic models of the chaotic systems that are used. The most popular mathematical models used in the literature on control of chaos are represented by the systems of ordinary differential equations. In some works we can also find discrete models defined by difference state equations. The second important thing is the formulation of the problems of control of chaotic processes. An important type of problems of control of chaotic processes is represented by the modification of the attractors, for example, transformation of chaotic oscillations into periodic and so on.

2. FRACTIONAL-ORDER NONLINEAR SYSTEMS

In this paper, we will consider the general incommensurate fractional-order nonlinear system represented as follows:

$$\begin{aligned} {}_0D_t^{q_i} x_i(t) &= f_i(x_1(t), x_2(t), \dots, x_n(t), t) \\ x_i(0) &= c_i, \quad i = 1, 2, \dots, n, \end{aligned} \qquad (1)$$

where c_i are initial conditions. The vector form of (1) is:

$$D^q x = f(x), \qquad (2)$$

where $q = [q_1, q_2, \dots, q_n]^T$ for $0 < q_i < 2$, ($i = 1, 2, \dots, n$) and $x \in R^n$, and where ${}_0D_t^q$ is the Caputo's derivative.

The Caputo's definition of fractional derivatives can be written as (Podlubny, 1999):

$${}_aD_t^q f(t) = \frac{1}{\Gamma(m-q)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q-m+1}} d\tau, \quad (m-1 < q < m) \qquad (3)$$

In Eq. (3) we assume boundary $a = 0$. Other definitions of the fractional derivative can be found in (Podlubny, 1999).

3. SYNCHRONIZATION OF CHAOTIC SYSTEMS

The important class of the control objectives corresponds to the problems of synchronization or, more precisely, controllable synchronization as opposed to the autosynchronization. Numerous publications on control of synchronization of chaotic processes and its application in the data transmission systems appeared in the 1990's. In the general case, by the synchronization is meant the coordinated variation of the states of two or more systems or, possibly, coordinated variation of some of their characteristics such as oscillation frequencies.

Let us take a look at the synchronization more closely. Several methods can be used for synchronization of chaotic systems. In this paragraph we will mention three well-known methods. If chaos synchronization is achieved by drive-response systems, the instability measure is negative. That means the system is not chaotic.

The first method is the Master-Slave (or drive-response) configuration scheme of two autonomous-dimensional fractional-order chaotic systems (Lu, 2005; Peng, 2007):

$$\begin{cases} M: & \dfrac{d^\alpha x}{dt^\alpha} = f(x), \\ S: & \dfrac{d^\alpha \tilde{x}}{dt^\alpha} = f(\tilde{x}) + C(x - \tilde{x}), \end{cases} \qquad (4)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$, $\alpha_i > 0$, is the fractional order and the systems are chaotic. C is a coupling matrix. For simplicity, let C have the form: $C = diag(d_1, d_2, \ldots, d_n)$, where $d_i \geq 0$. The error is $e = x - \tilde{x}$ and the aim of the synchroniziation is to design the coupling matrix such that $\| e(t) \| \to 0$ as $t \to +\infty$.

The second method is the method for constructing the drive-response configuration, which was introduced by Pecora and Carroll in 1990, known as a (PC). Let us build a PC drive-response configuration in which a drive system is given by the fractional-order system (with three state variables: *x, y, z*) and a response system is given by the subspace containing the (*x, y*) variables. Then we can use the chaotic signal *z* to drive the response subsystem.

The third method is the synchronization via active-passive decomposition method (APD). Let us build an APD drive-response configuration with a drive system given by the chaotic system and with a response system given by its replica. Then we can take *s*(*t*) as a drive signal (Li et al., 2006).

Chaos synchronization and its potential application to secure communications have attracted much attention from various disciplines in science and engineering since the pioneering work of (Pecora and Carroll, 1990). In this section, we briefly discuss the chaos synchronization methods between the chaotic fractional-order systems and we can also mention method via master-slave configuration with linear coupling (Zhu et al., 2009).

4. CONTROL OF CHAOTIC SYSTEMS

In (Andrievskii and Fradkov, 2003, 2004) were collected and presented several methods used for the control of chaotic processes. The authors considered the classical integer-order chaotic systems but in general we can use those methods for the fractional-order chaotic systems as well. In addition some other methods have been proposed for control of such systems and they can be summarized as follows (Petráš, 2011):

1. Open loop (feed-forward) control is based on varying behavior of the nonlinear system under the action of predetermined external input. This approach is simple because it does without any measurements or sensors. This is especially important for the control of superfast processes.
2. Linear and nonlinear (feed-back) control deals with the possibilities of using the traditional approaches, and methods of automatic control to the problems of chaos control are discussed in numerous papers. The desired aim can be reached sometimes even by means of the simple proportional law of control and feedback. The potentialities of the dynamic feedbacks can be better realized by using the observers. Other methods of the modern theory of nonlinear control such as the theory of center manifold, sliding mode control, the backstepping procedure, the reset control, the H∞-optimal design and so on can be used to solve the problems of stabilization about the given state.
3. Adaptive control assumes the possibility of applying the methods of adaptation to the control of chaotic processes, where the parameters of the controlled plant are unknown and the information about the model structure more often than not is incomplete. A number of the existing methods of adaptation such as the methods of gradient and speed gradient, least squares, maximum likelihood, and so on can be used to develop algorithms of adaptive control and parametric identification. A controller is usually designed using the reference model or the methods of linearization by feedback.
4. Linearization of the Poincaré map method can be formulated by the following two key ideas: (i) designing controller by the discrete system model based on linearization of the Poincaré map and (ii) using the property of recurrence of the chaotic trajectories and applying the control action only at the instants when the trajectory returns to some neighborhood of the desired state or given orbit.
5. Time-delayed feedback method considers the problem of stabilizing an unstable periodic orbit of a nonlinear system by a simple feedback law with time delay. Sensitivity to the parameter, especially to the delay time, is a disadvantage of the control law.
6. Neural network-based control deals with the ability of neural networks to control and predict behavior of nonlinear systems. The various structures of neural networks for control and prediction of the processes in nonlinear chaotic systems can be found in literature.
7. Fuzzy control uses a description of system indeterminacy in terms of fuzzy models, provides specific versions of the control algorithms, which consists of four blocks: knowledge base, fuzzification, inference and defuzzification.

5. NEW CHAOS CONTROL STRATEGY

The fractional calculus techniques as for example a fractional differentiator based controller of a fractional integrator based controller can also be used (Tavazoei et al., 2009). Both of them are particular cases of the fractional-order controllers described as (Podlubny, 1999):

$$u(t) = K_p e(t) + T_i \, {}_0D_t^{-\lambda} e(t) + T_d \, {}_0D_t^{\delta} e(t), \quad (\lambda, \delta > 0), \qquad (5)$$

where K_p is the proportional constant, T_i is the integration constant and T_d is the differentiation constant. Controller (5) is more flexible than classical one and gives better results of the control performances (Monje et al., 2010).

6. EXAMPLE: SLIDING MODE CONTROL OF THE FRACTIONAL-ORDER ECONOMICAL SYSTEM

A sliding model control (SMC) strategy is also applicable for the fractional-order chaotic systems. It is a form of variable structure control method that alters the dynamics of a nonlinear system by application of a high-frequency switching control. The state feedback control law is not a continuous function of time. It switches from one continuous structure to another based on the current position in the state space. Trajectories always move toward a switching condition. The motion of the system as it slides along these boundaries is called a sliding mode. The sliding mode control scheme involves: (i) selection of the sliding surface that represents a desirable system dynamic behavior, (ii) finding a switching control law that a sliding mode exists on every point of the sliding surface.

Consider the following general structure of the fractional-order nonlinear system under control

$$ {}_0D_t^q x(t) = f(x(t)) + Bu(t), \tag{6} $$

where $u(t) = [u_1(t)u_2(t)...u_m(t)]^T$ is an m-dimensional input vector that will be used and the following control structure will be considered for state feedback:

$$ u(t) = u_{eq}(t) + u_{sw}(t), \tag{7} $$

where $u_{eq}(t)$ is the equivalent control and $u_{sw}(t)$ is the switching control of the system (6). A common task is to design a state feedback control law to stabilize the dynamical system (6) around the origin $x(t) = [0, 0, \ldots, 0]^T$. In the sliding mode, the sliding surface and its derivative must satisfy $\sigma(t) = 0$ and $\dot{\sigma}(t) = 0$.

Let us use the controlled fractional-order financial system in the form (Dadras and Momeni, 2010):

$$ \begin{aligned} {}_0D_t^{q_1} x_1(t) &= x_3(t) + (x_2(t) - a)x_1(t), \\ {}_0D_t^{q_2} x_2(t) &= 1 - bx_2(t) - x_1^2(t) + u(t), \\ {}_0D_t^{q_3} x_3(t) &= -x_1(t) - cx_3(t), \end{aligned} \tag{8} $$

where a is the saving amount, b is the cost per investment, and c is the elasticity of demand of commercial market, $(a,b,c) \in R$ and $(a,b,c) > 0$. The state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$ are the interest rate, the investment demand, and the price index, respectively.

The proposed fractional sliding surface is defined as

$$ \sigma(t) = \int_0^t (x_1^2(\tau) + Kx_2(\tau))d\tau + {}_0D_t^{q_2-1} x_2(t), \tag{9} $$

where K is a positive constant, in addition $K = K_{eq}$. The equivalent control $u_{eq}(t)$ is obtained by setting the derivative of sliding surface to zero and then solving the second equation of (8) for $u(t)$. We obtain

$$ {}_0D_t^{q_2} x_2(t) = -(x_1^2(t) + Kx_2(t)) $$

and then we get the relation

$$ \begin{aligned} u_{eq}(t) &= {}_0D_t^{q_2} x_2(t) - 1 + bx_2(t) + x_1^2(t) \\ &= -(x_1^2(t) + K_{eq}x_2(t) - 1 + bx_2(t) + x_1^2(t) \\ &= (b - K_{eq})x_2(t) - 1, \end{aligned} \tag{10} $$

where K_{eq} is the constant of the controller.

The switching control $u_{sw}(t)$ law is chosen in order to satisfy the sliding condition

$$ u_{sw}(t) = K_{sw}\, sign(\sigma), \tag{11} $$

where

$$ sign(\sigma) = \begin{cases} +1 & : \sigma > 0, \\ 0 & : \sigma = 0, \\ -1 & : \sigma < 0, \end{cases} $$

and K_{sw} is the gain of the controller ($K_{sw} < 0$). Finally, the total control law is defined as follows:

$$ u(t) = u_{eq}(t) + u_{sw}(t) = (b - K_{eq})x_2(t) - 1 + K_{sw}\, sign(\sigma). \tag{12} $$

We assume the following parameters of the chaotic system (8): $a = 1$, $b = 0.1$, $c = 1$, and the controller (12) parameters, experimentally found: $K_{eq} = 1.5$ and $K_{sw} = -3.5$. The controller will be applied at $t = 30$ *s*. In the first case we use a commensurate order of derivatives $q_1 = q_2 = q_3 = 0.9$ and in the second case we use an incommensurate order of the derivatives $q_1 = 1.0$, $q_2 = 0.95$, and $q_3 = 0.99$ of the fractional-order chaotic system (8). The initial conditions for both cases are $(x_1(0), x_2(0), x_3(0)) = (2, -1, 1)$.

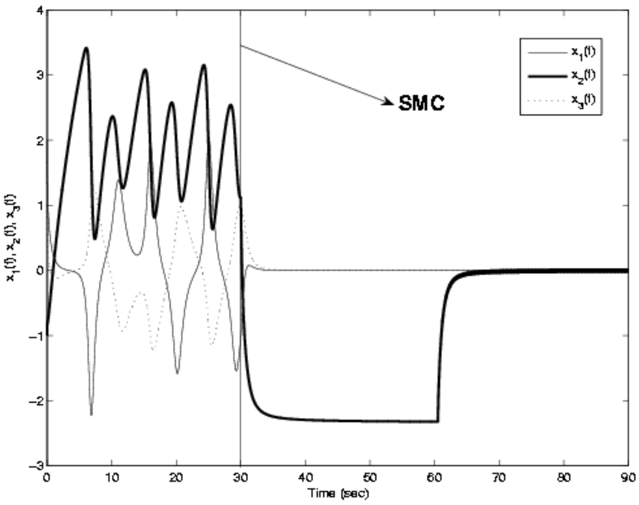

Fig. 1. Controlled state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$ of commensurate fractional-order financial system, where the SMC was activated at 30 *s*

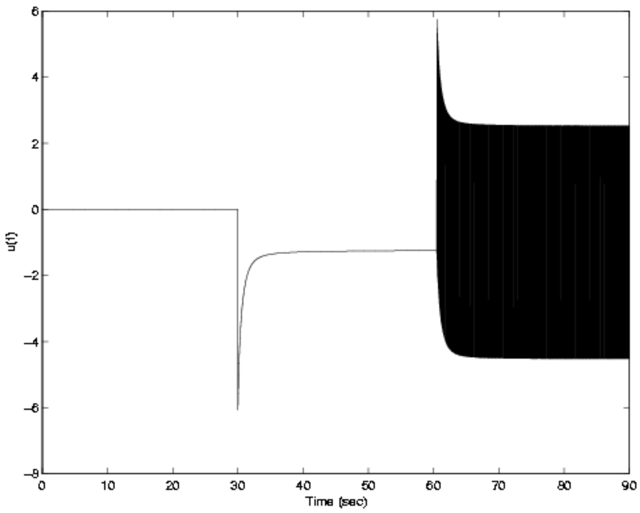

Fig. 2. Time response of control law $u(t)$ for commensurate fractional-order system

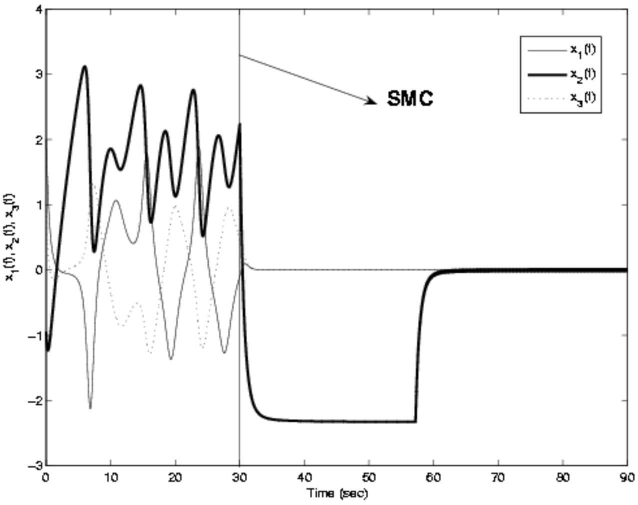

Fig. 3. Controlled state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$ of incommensurate fractional-order financial system, where the SMC was activated at 30 *s*

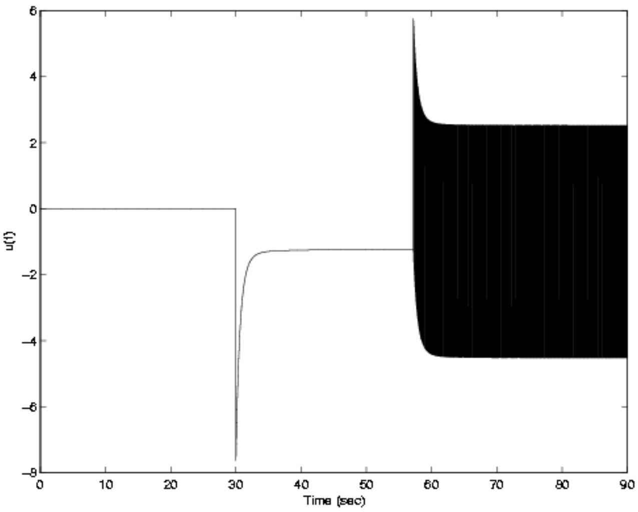

Fig. 4. Time response of control law $u(t)$ for incommensurate fractional-order system

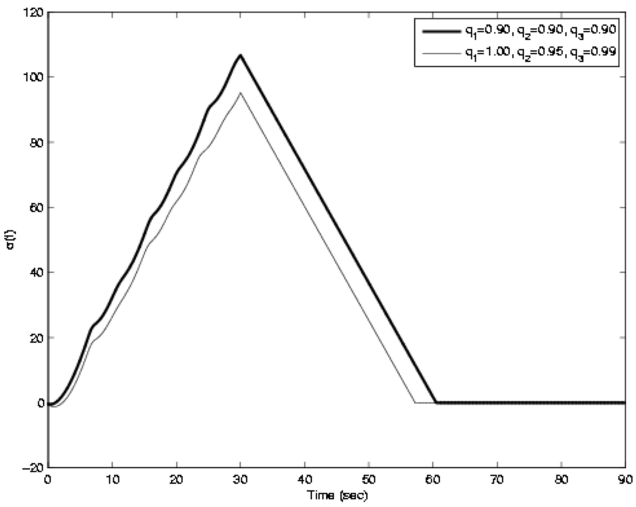

Fig. 5. Time responses of sliding surfaces $\sigma(t)$

In Fig. 1 are depicted the controlled state variables of the commensurate fractional-order financial systems (8) with the parameters: $a = 1$, $b = 0.1$, $c = 1$, orders $q_1 = q_2 = q_3 = 0.9$, controller (12) parameters: $K_{eq} = 1.5$ and $K_{sw} = -3.5$, initial conditions: $(x_1(0), x_2(0), x_3(0)) = (2, -1, 1)$ for simulation time $T_{sim} = 90$ *s* and time step $h = 0.005$.

In Fig. 2 is shown the control law of commensurate fractional-order financial system which drives the system states to the sliding surface. We can observe chattering in the sliding mode.

In Fig. 3 are depicted the controlled state variables of the incommensurate fractional-order financial systems (8) with the parameters: $a = 1$, $b = 0.1$, $c = 1$, orders $q_1 = 1.0$, $q_2 = 0.95$, and $q_3 = 0.99$, controller (12) parameters: $K_{eq} = 1.5$ and $K_{sw} = -3.5$, initial conditions: $(x_1(0), x_2(0), x_3(0)) = (2, -1, 1)$ for simulation time $T_{sim} = 90$ *s* and time step $h = 0.005$.

In Fig. 4 is shown the control law of incommensurate fractional-order financial system which drives the system states to the sliding surface. We can again observe chattering in the sliding mode.

In Fig. 5 are depicted the time responses of the sliding surface. We can observe that the controller kept the system states on the sliding surface for all subsequent time.

Performed simulations show that system responses after applying the control law (12) are satisfactory for both cases. The results confirm that obtained control strategy is efficient for controlling the fractional-order financial system (8) for various parameters (Petráš and Bednárová, 2010).

7. CONCLUSIONS

In this article is presented a review of the control strategies for the fractional-order nonlinear systems. On illustrative example is shown the SMC control method. This control method is simple and control law achieved asymptotically stabilized system if the controller is applied to the investment demand in order to control the whole economical system. This approach is applicable for different types of the fractional-order chaotic systems as well as the other control strategies (Monje et al., 2010).

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