VARIABLE-, FRACTIONAL-ORDERS CLOSED-LOOP SYSTEMS DESCRIPTION

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Abstract: In this paper we explore the linear difference equations with fractional orders, which are functions of time. A description of closed-loop dynamical systems described by such equations is proposed. In a numerical example a simple control strategy based on time-varying fractional orders is presented.

1. INTRODUCTION

Over the last few decades a growing interest in a fractional calculus (Carpinteri and Mainardi, 1997; Lubich, 1986; Miller and Ross, 1993; Oustaloup, 1995; Podlubny, 1999) has been observed. Its technical application effected in new fractional-order models of physical processes and materials behaviour (Oustaloup, 1995; Podlubny, 1999). Extensive possibilities of modern digital processors to analyse, modify or extract information from measured signals require describing signals by discrete-time functions (Ostalczyk, 2001). In practical applications the use of a backward difference is necessary.

A variable- (V), fractional order (FO) backward difference (BD) of a discrete-time bounded function f_k is defined as follows (Ostalczyk, 2000, 2003, Ostalczyk and Derkacz, 2003).

$$ {}_0\Delta_{\infty}^{(n_j)} f_k = \sum_{i=0}^{\infty} b_i^{(n_j)} f_{k-i} = \begin{bmatrix} b_0^{(n_j)} & b_1^{(n_j)} & \cdots & b_k^{(n_j)} & b_{k+1}^{(n_j)} & b_{k+2}^{(n_j)} & \cdots \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_1 \\ f_0 \\ f_{-1} \\ \vdots \end{bmatrix}, \quad (1) $$

where a difference order $n_j \in \mathbf{R}$ and discrete time instants $k \in \mathbf{N} \cup \{0\}$ ($\mathbf{R}$ and $\mathbf{N}$ denote sets of real and natural numbers, respectively). Coefficients $b_i^{(n_j)}$ are defined below

$$ b_i^{(n_j)} = \begin{cases} 1 & \text{for} \quad i=0 \\ \dfrac{n_j(n_j-1)\cdots(n_j-i+1)}{i!} & \text{for} \quad i=1,2,3\cdots \end{cases}. \quad (2) $$

For discrete-time functions f_k satisfying $f_k = 0$ for k<0 we can write

$$ {}_0\Delta_{k}^{(n_j)} f_k = \begin{bmatrix} b_0^{(n_j)} & b_1^{(n_j)} & \cdots & b_{k-1}^{(n_j)} & b_k^{(n_j)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_1 \\ f_0 \end{bmatrix}. \quad (3) $$

Realize that in the formula given above the constant order n_j is independent of the time-variable k. Next we define a discrete variable function

$$ \mathbf{N} \cup \{0\} \ni j \to n_j \in \mathbf{R}. \quad (4) $$

The VFOBD defined by formula (3) is a function of two discrete variables k and j

$$ \mathbf{N} \cup \{0\} \ni k \to g_{k,j} = {}_0\Delta_k^{(n_j)} f_k \in \mathbf{R}. \quad (5) $$

For a special assignment of n_j and k we define a new one discrete variable function

$$ \mathbf{N} \cup \{0\} \ni k \to h_k = {}_0\Delta_k^{(n_k)} f_k \in \mathbf{R}, \quad (6) $$

defined as

$$ h_k = {}_0\Delta_{\infty}^{(n_k)} f_k = \sum_{i=0}^{\infty} b_i^{(n_k)} f_{k-i} = \begin{bmatrix} b_0^{(n_k)} & b_1^{(n_k)} & \cdots & b_k^{(n_k)} & b_{k+1}^{(n_k)} & b_{k+2}^{(n_k)} & \cdots \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_1 \\ f_0 \\ f_{-1} \\ \vdots \end{bmatrix} \quad (7) $$

with

$$ b_i^{(n_k)} = \begin{cases} 1 & \text{for} \quad i=0 \\ \dfrac{n_k(n_k-1)\cdots(n_k-i+1)}{i!} & \text{for} \quad i=1,2,3\cdots \end{cases}. \quad (8) $$

For $f_k = 0$ for $k<0$ we obtain

$$ {}_0\Delta_k^{(n_k)} f_k = \sum_{i=0}^{k} b_i^{(n_k)} f_{k-i}. \quad (9) $$

It should be noted that the VFOBD is related to a variable-order fractional operator defined as (Coimbra, 2003; Lorenzo and Hartley, 2002)

$$ {}_0d_t^{n(t)}y(t) = \int_0^t \frac{(t-\tau)^{-n(t)-1}}{\Gamma(-n(t))} y(\tau)d\tau . \quad (10)$$

It can be proved (Oustaloup, 1995) that for every $t \geq 0$ the formula given above is equivalent to the following limit

$$ {}_0d_t^{n(t)}y(t) = \lim_{h\to 0} \frac{\sum_{i=0}^{\left[\frac{t}{h}\right]} b_i^{(n(t))} y(t-ih)}{h^{n(t)}} . \quad (11)$$

where [.] denotes a function rounding towards the nearest integer, $h > 0$ is an integration step. Substituting $t = kh$ with $h = 1$ we immediately obtain a VFOBD defined by formula (9)

In this paper we focus our attention on linear systems described by VFOBD equations with time-invariant coefficients. Next we explore an adequate description of a closed-loop system with a controller and plant modelled by a FO difference equation. In the second numerical example we show that even though the physical processes described by VFOBD equations are yet unknown, they are useful in a control strategies design.

2. LINEAR VFO DISCRETE-TIME SYSTEMS

Now we consider a linear VFO difference equation (DE)

$$\sum_{i=0}^{r} A_i\, {}_0\Delta_\infty^{(n_{i,k})} y_k = \sum_{j=0}^{s} B_j\, {}_0\Delta_\infty^{(m_{j,k})} u_k , \quad (12)$$

where

$$A_r \neq 0 ,\ A_i, B_j \in \mathbf{R} ,\ i = 0,1,2,\cdots,r ,\ j = 0,1,2,\cdots,s , \quad (13)$$

$$n_{rk}, n_{r-1,k}, \cdots, n_{1,k}, n_{0,k} \in \mathbf{R} \quad (14)$$

$$m_{s,k}, m_{s-1,k}, \cdots, m_{1,k}, m_{0,k} \in \mathbf{R}$$

Here we admit a case when some $n_{i,k} = n_{j,k}$, $m_{i,k} = m_{j,k}$ or even $n_{i,k} = n_{j,k} = 0$ for $i \neq j$ (a subscript k denotes an appropriate discrete time instant).

From this point on we will make use of a permanent assumption that $u_k = 0$ for $k < 0$. Hence difference Equation (12) can be expressed in the following form

$$\begin{bmatrix} A_r & A_{r-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} {}_0\Delta_\infty^{(n_{r,k})} y_k \\ {}_0\Delta_\infty^{(n_{r-1,k})} y_k \\ \vdots \\ {}_0\Delta_\infty^{(n_{1,k})} y_k \\ {}_0\Delta_\infty^{(n_{0,k})} y_k \end{bmatrix} = \quad (15)$$

$$= \begin{bmatrix} B_s & B_{s-1} & \cdots & B_1 & B_0 \end{bmatrix} \begin{bmatrix} {}_0\Delta_k^{(m_{s,k})} u_k \\ {}_0\Delta_k^{(m_{s-1,k})} u_k \\ \vdots \\ {}_0\Delta_k^{(m_{1,k})} u_k \\ {}_0\Delta_k^{(m_{0,k})} u_k \end{bmatrix} .$$

Substituting the column vectors in difference equation (15) by formulae (1) and (3) yields

$$\begin{bmatrix} A_r & A_{r-1} & \cdots & A_1 & A_0 \end{bmatrix} \times \quad (16)$$

$$\times \begin{bmatrix} b_0^{(n_{r,k})} & b_1^{(n_{r,k})} & \cdots & b_k^{(n_{r,k})} & \cdots \\ b_0^{(n_{r-1,k})} & b_1^{(n_{r-1,k})} & \cdots & b_k^{(n_{r-1,k})} & \cdots \\ \vdots & \vdots & & \vdots & \\ b_0^{(n_{1,k})} & b_1^{(n_{1,k})} & \cdots & b_k^{(n_{1,k})} & \cdots \\ b_0^{(n_{0,k})} & b_1^{(n_{0,k})} & \cdots & b_k^{(n_{0,k})} & \cdots \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_0 \\ \vdots \end{bmatrix} =$$

$$= \begin{bmatrix} B_s & B_{s-1} & \cdots & B_1 & B_0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} b_0^{(m_{s,k})} & b_1^{(m_{s,k})} & \cdots & b_{k-1}^{(m_{s,k})} & b_k^{(m_{s,k})} \\ b_0^{(m_{s-1,k})} & b_1^{(m_{s-1,k})} & \cdots & b_{k-1}^{(m_{s-1,k})} & b_k^{(m_{s-1,k})} \\ \vdots & \vdots & & \vdots & \vdots \\ b_0^{(m_{1,k})} & b_1^{(m_{1,k})} & \cdots & b_{k-1}^{(m_{1,k})} & b_k^{(m_{1,k})} \\ b_0^{(m_{0,k})} & b_1^{(m_{0,k})} & \cdots & b_{k-1}^{(m_{0,k})} & b_k^{(m_{0,k})} \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} .$$

Multiplications of the row-vectors by appropriate matrices in Equation (16) yield

$$\begin{bmatrix} a_{0,k} & a_{1,k} & \cdots & a_{k,k} & \cdots \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_0 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_{0,k} & \cdots & b_{k,k} \end{bmatrix} \begin{bmatrix} u_k \\ \vdots \\ u_0 \end{bmatrix} , \quad (17)$$

where

$$a_{i,k} = \begin{bmatrix} A_r & A_{r-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} b_i^{(n_{r,k})} \\ b_i^{(n_{r-1,k})} \\ \vdots \\ b_i^{(n_{1,k})} \\ b_i^{(n_{0,k})} \end{bmatrix} ,\ i = 0,1,2,\cdots , \quad (18)$$

and

$$b_{j,k} = \begin{bmatrix} B_s & B_{s-1} & \cdots & B_1 & B_0 \end{bmatrix} \begin{bmatrix} b_j^{(m_{s,k})} \\ b_j^{(m_{s-1,k})} \\ \vdots \\ b_j^{(m_{1,k})} \\ b_j^{(m_{0,k})} \end{bmatrix} ,\ j = 0,1,,\cdots,k .$$

Further we assume that $a_{0,k} \neq 0$ for all k. It should be noted that practical realisations of the discrete-time systems impose an additional condition $\max\{n_{r,k}, \dots, n_{0,k}\} \geq \max\{m_{s,k}, \dots, m_{0,k}\}$ on all non-negative k. Equation (17) is valid for every positive integer. Thus for $k - 1$ we have

$$\begin{bmatrix} a_{0,k} & a_{1,k} & \cdots & a_{k,k} & \cdots \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_0 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_{0,k} & \cdots & b_{k,k} \end{bmatrix} \begin{bmatrix} u_k \\ \vdots \\ u_0 \end{bmatrix}, \tag{19}$$

while

$$\begin{bmatrix} a_{0,k-1} & a_{1,k-1} & \cdots & a_{k-1,k-1} & \cdots \end{bmatrix} \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_0 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_{0,k-1} & \cdots & b_{k-1,k-1} \end{bmatrix} \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix} \tag{20}$$

This can be further transformed into an equivalent form

$$\begin{bmatrix} 0 & a_{0,k-1} & \cdots & a_{k-1,k-1} \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_0 \end{bmatrix} + \begin{bmatrix} a_{k,k-1} & a_{k+1,k-1} & \cdots \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \\ \vdots \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & b_{0,k-1} & \cdots & b_{k-2,k-1} & b_{k-1,k-1} \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}. \tag{21}$$

Repeating this notation for $k-1, k-2, \dots, 1, 0$ and putting them together in the matrix-vector form we get

$$\begin{bmatrix} a_{0,k} & a_{1,k} & \cdots & a_{k-1,k} & a_{k,k} \\ 0 & a_{0,k-1} & \cdots & a_{k-2,k-1} & a_{k-1,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_{0,1} & a_{1,1} \\ 0 & 0 & \cdots & 0 & a_{0,0} \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} +$$

$$+ \begin{bmatrix} a_{k+1,k} & a_{k+2,k} & a_{k+3,k} & \cdots \\ a_{k,k-1} & a_{k+1,k-1} & a_{k+2,k-1} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{1,0} & a_{2,0} & a_{3,0} & \cdots \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \\ \vdots \end{bmatrix} = \tag{22}$$

$$= \begin{bmatrix} b_{0,k} & b_{1,0} & \cdots & b_{k-1,k} & b_{k,k} \\ 0 & b_{0,k-1} & \cdots & b_{k-2,k-1} & b_{k-1,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & b_{0,1} & b_{1,1} \\ 0 & 0 & \cdots & 0 & b_{0,0} \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}$$

or

$$\mathbf{D}_k \mathbf{y}_k + \mathbf{I}_k \mathbf{y}_I = \mathbf{N}_k \mathbf{u}_k, \tag{23}$$

where

$$\mathbf{D}_k = \begin{bmatrix} a_{0,k} & a_{1,k} & \cdots & a_{k-1,k} & a_{k,k} \\ 0 & a_{0,k-1} & \cdots & a_{k-2,k-1} & a_{k-1,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_{0,1} & a_{1,1} \\ 0 & 0 & \cdots & 0 & a_{0,0} \end{bmatrix}, \tag{24}$$

is $(k+1)\times(k+1)$ output matrix,

$$\mathbf{I}_k = \begin{bmatrix} a_{k+1,k} & a_{k+2,k} & a_{k+3,k} & \cdots \\ a_{k,k-1} & a_{k+1,k-1} & a_{k+2,k-1} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{1,0} & a_{2,0} & a_{3,0} & \cdots \end{bmatrix}, \tag{25}$$

is $(k+1)\times(\infty)$ initial conditions matrix,

$$\mathbf{N}_k = \begin{bmatrix} b_{0,k} & b_{1,0} & \cdots & b_{k-1,k} & b_{k,k} \\ 0 & b_{0,k-1} & \cdots & b_{k-2,k-1} & b_{k-1,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & b_{0,1} & b_{1,1} \\ 0 & 0 & \cdots & 0 & b_{0,0} \end{bmatrix}, \tag{26}$$

is $(k+1)\times(k+1)$ input matrix,

$$\mathbf{y}_k = \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix}, \mathbf{y}_I = \begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \\ \vdots \end{bmatrix}, \mathbf{u}_k = \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}, \tag{27}$$

are $(k+1)\times 1$ output, $\infty \times 1$ initial conditions, and $(k+1)\times 1$ input vectors, respectively. Square matrix (24) is always non-singular. Hence Equation (23) can be always rearranged into the form

$$\mathbf{y}_k = \mathbf{D}_k^{-1}\mathbf{N}_k\mathbf{u}_k - \mathbf{D}_k^{-1}\mathbf{I}_k\mathbf{y}_I, \tag{28}$$

where the first right-hand side term denotes a forced part of the response, the second a homogenous one. The above investigations are illustrated by following numerical example.

2.1. Numerical example

Consider the VFODE of the form

$${}_0\Delta_\infty^{(n_{1,k})} y_k + \frac{1}{2} y_k = \frac{1}{2} u_k, \tag{29}$$

For a given external function $u_k = 1_{k-1}$ (discrete unit step function) and the assumed zero initial conditions $0 = y_{-1}, y_{-2}, y_{-3} = \cdots$ one should find an order function $n_{1,k}$ for which the solution has the form

$$y_k = \begin{cases} \alpha k & \text{for} \quad 0 \le k \le k_s \\ 1 & \text{for} \quad k_s < k \end{cases}, \tag{30}$$

where α and k_s mean undetermined yet: the response slope parameter and the switch time instant, respectively. Substituting (3) into (29) we get (15) and further (17). For $k-1, k-2, \dots, 1, 0$ we obtain (23) with

$$\mathbf{y}_k = \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix}, \mathbf{y}_I = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \mathbf{u}_k = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}, \tag{31}$$

$$\mathbf{D}_k = \begin{bmatrix} 1.5 & b_1^{(n_{1,k})} & \cdots & b_{k-1}^{(n_{1,k})} & b_k^{(n_{1,k})} \\ 0 & 1.5 & \cdots & b_{k-2}^{(n_{1,k-1})} & b_{k-1}^{(n_{1,k-1})} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1.5 & b_1^{(n_{1,1})} \\ 0 & 0 & \cdots & 0 & 1.5 \end{bmatrix}, \tag{32}$$

$$\mathbf{I}_k = \begin{bmatrix} b_{k+1}^{(n_{1,k})} & b_{k+2}^{(n_{1,k})} & b_{k+3}^{(n_{1,k})} & \cdots \\ b_{k+1}^{(n_{1,k-1})} & b_{k+2}^{(n_{1,k-1})} & b_{k+3}^{(n_{1,k-1})} & \cdots \\ \vdots & \vdots & \vdots & \\ b_1^{(n_{1,0})} & b_2^{(n_{1,0})} & b_3^{(n_{1,0})} & \cdots \end{bmatrix}, \tag{33}$$

$$\mathbf{N}_k = \begin{bmatrix} 0 & 0.5 & 0 & \cdots & 0 \\ 0 & 0 & 0.5 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{34}$$

Now we evaluate the consecutive values of an order function $n_{1,k}$. For $k=0$ and any $n_{1,0}$ the unique solution of equation (29) is $y_0 = 0$. For $k=1$ and any $n_{1,1}$ one possible solution is $y_1 = 1/3$. Hence we must put $\alpha = 1/3$. This implies that $y_2 = 2/3$ and $y_i = 1$ for $i = 3, 4, \cdots$. Further, for $k=2$, from formulae (28) and (31) – (36) we obtain

$$1.5y_2 - n_{1,2}y_1 + 0.5n_{1,2}(n_{1,2}-1)y_0 = 0.5\,, \tag{35}$$

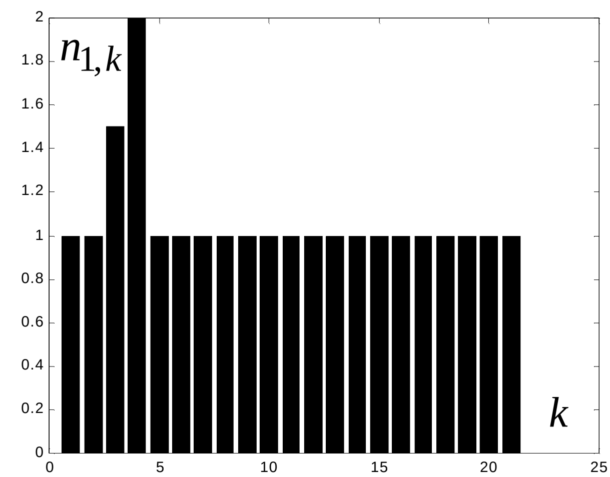

Fig. 1. Plot of an order function $n_{1,k}$

Substituting appropriate values, after elementary operations, we get $n_{1,2} = 1{,}5$. For $k=3$ we get equation

$$1.5y_3 - n_{1,3}y_2 + \frac{n_{1,3}(n_{1,3}-1)}{2}y_1 + \frac{n_{1,3}(n_{1,3}-1)(n_{1,3}-2)}{6}y_0 = 0.5\,, \tag{36}$$

From two possible solutions $n_{1,3} = 2$ and $n_{1,3} = 3$ we take the first one. Continuing this procedure as an order function we take

$$n_{1,k} = \begin{cases} 1\,(\text{any}) & \text{for} \quad k = 0,1 \\ 1.5 & \text{for} \quad k = 2 \\ 2 & \text{for} \quad k = 3 \\ 1 & \text{for} \quad k \ge 4 \end{cases}, \tag{37}$$

Its plot is presented in Fig. 1 .

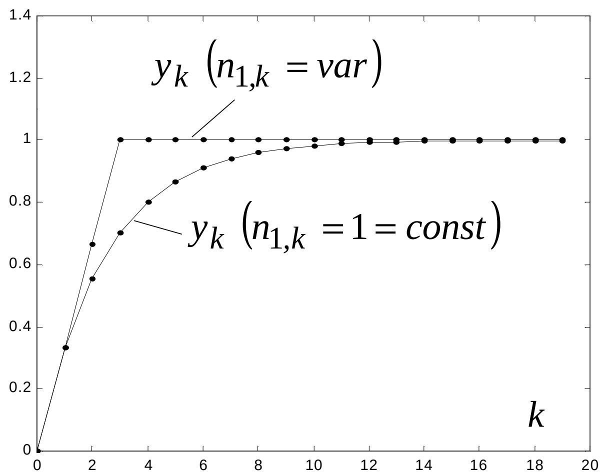

Fig. 2. Plot of the VFODE and classical first-order DE solutions

The solution to difference equation (29) with order function (37) is plotted in Fig. 2. Here, for the sake of comparison, the solution to a classical fist-order difference equation ($n_{1,k} = 1 = \text{const}$) is also plotted.

The numerical example considered above shows that it is possible to reshape the solution to a VFODE. It should be noted that although y_0, y_1 do not depend on $n_{1,0}, n_{1,1}$ for $k \ge 2$ we have

$$\begin{matrix} y_2(n_{1,2}) \\ y_3(n_{1,2}, n_{1,3}) \\ y_4(n_{1,2}, n_{1,3}, n_{1,4}) \\ \vdots \end{matrix}, \tag{38}$$

3. DESCRIPTION OF A CLOSED - LOOP DYNAMICAL SYSTEM

Next we consider a unity-feedback system with a linear discrete-time plant and a discrete-time controller. In general, we assume that a time-invariant coefficients plant is described by the (time-invariant or time-variant) fractional order difference equation. A block diagram of a closed-loop system is presented in Fig. 3.

A plant is described by an equation similar to Equation (28)

$$\mathbf{p}_k = \mathbf{D}_{P,k}^{-1}\mathbf{N}_{P,k}\mathbf{u}_k - \mathbf{D}_{P,k}^{-1}\mathbf{I}_{P,k}\mathbf{p}_I \tag{39}$$

where $\mathbf{u}_k, \mathbf{p}_k$ are the plant input and output signals, respectively. The vectors $\mathbf{p}_I$ and $\mathbf{v}_I$ denote the plant and regulator initial conditions, respectively. The plant is controlled by the controller output signal $\mathbf{v}_k$ and subjected to a plant disturbance signal $\mathbf{d}_{i,k}$ vector. A controller algorithm is described by VFODE (12) or equivalently by the matrix-vector Eq. (28)

$$\mathbf{v}_k = \mathbf{D}_{R,k}^{-1}\mathbf{N}_{R,k}\mathbf{e}_k - \mathbf{D}_{R,k}^{-1}\mathbf{I}_{R,k}\mathbf{v}_I \tag{40}$$

where $\mathbf{e}_k$ denotes the controller input (a closed-loop system error). Column vectors $\mathbf{r}_k, \mathbf{d}_k$, $\mathbf{n}_k$ denote: a system command, a plant output disturbance and the sensor noise signals, respectively. The vector $\mathbf{y}_k$ is a system output signal. Additional four equations describe the closed-loop system

$$\mathbf{e}_k = \mathbf{r}_k - \mathbf{w}_k\,,\ \mathbf{w}_k = \mathbf{y}_k + \mathbf{n}_k\,,$$
$$\mathbf{y}_k = \mathbf{p}_k + \mathbf{d}_k\,, \mathbf{u}_k = \mathbf{v}_k + \mathbf{d}_{ik}\,. \tag{41}$$

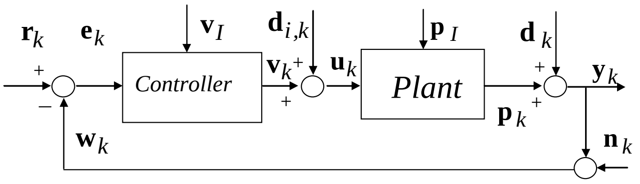

Fig. 3. Block diagram of the closed-loop system

Combining Equations (39)-(41) we get an input/output description of the closed-loop system

$$\mathbf{y}_k = \left(\mathbf{1}_k + \mathbf{G}_{O,k}\right)^{-1}\left(\mathbf{G}_{O,k}\mathbf{r}_k - \mathbf{G}_{O,k}\mathbf{n}_k + \mathbf{d}_k\right) + \left(\mathbf{1}_k + \mathbf{G}_{O,k}\right)^{-1}\mathbf{D}_{P,k}^{-1}\mathbf{N}_{P,k}\mathbf{d}_{i,k} - \left(\mathbf{1}_k + \mathbf{G}_{O,k}\right)^{-1}\mathbf{D}_{P,k}^{-1}\left[\mathbf{N}_{P,k}\mathbf{D}_{R,k}^{-1}\mathbf{I}_{R,k} \quad \mathbf{I}_{P,k}\right]\begin{bmatrix}\mathbf{v}_I\\ \mathbf{p}_I\end{bmatrix} \tag{42}$$

where $\mathbf{G}_{O,k} = \mathbf{D}_{P,k}^{-1}\mathbf{N}_{P,k}\mathbf{D}_{R,k}^{-1}\mathbf{N}_{R,k}$ is an open loop system description, the matrix $\mathbf{1}_k$ is $(k+1)\times(k+1)$ unit matrix.

4. VFOs $\mathbf{PI^{(m_{1,k})}D^{(m_{2,k})}}$ CONTROLLER DESCRIPTION

Linear control strategies in the form of PID algorithms are still basic in digital control since they give satisfactory solutions to different control problems. In such controllers control strategies are implemented by software thus a realisation may be restricted mainly by a micro-controller memory and speed.

The constant fractional-order discrete-time PID controllers have been the subject of investigations for many years (Machado, 2001; Podlubny, 1999). Here we define a VFOs discrete-time $\mathrm{PI}^{(m_{1,k})}\mathrm{D}^{(m_{2,k})}$ controller. Its algorithm is described by a special case of Equation (12) where we put

$$r = 0, n_{0,k} = 0\,,\ s = 2, m_{2,k}, m_{1,k}, m_{0,k} \in \mathbf{R} \tag{43}$$

In general, to preserve the PID strategy, in general, we assume that $m_{0,k} = 0$ and $m_{1,k} > 0, m_{2,k} > 0$. According to Fig.3, the controller input is denoted by e_k and the output by v_k. Hence the $\mathrm{PI}^{(m_{1,k})}\mathrm{D}^{(m_{2,k})}$ controller is described by the following difference equation

$$v_k = \sum_{j=0}^{2} B_{j\,0}\Delta_\infty^{(m_{j,k})}e_k = \left[K_P \quad K_I \quad K_D\right]\begin{bmatrix} e_k \\ {}_0\Delta_\infty^{(m_{1,k})}e_k \\ {}_0\Delta_\infty^{(m_{2,k})}e_k \end{bmatrix} \tag{44}$$

where $B_0 = K_p$, $B_1 = K_I$, $B_2 = K_D$ denote proportional, integral and derivative gain, respectively. It is assumed that $K_p + K_I + K_D \neq 0$. Equation (44) implies that $\mathbf{D}_{R,k} = \mathbf{I}_k$, hence from (40) we get

$$\mathbf{v}_k = \mathbf{N}_{R,k}\mathbf{e}_k - \mathbf{I}_{R,k}\mathbf{v}_I\,, \tag{45}$$

where the matrix $\mathbf{N}_{R,k}$ is defined by equation (26) with

$$b_{j,k} = \left[K_P \quad K_I \quad K_D\right]\begin{bmatrix} b_j^{(m_{0,k})} \\ b_j^{(m_{1,k})} \\ b_j^{(m_{2,k})} \end{bmatrix},\ j = 0,1,2,\cdots,k-1,k\,. \tag{46}$$

The possible use of such a controller will be presented in the following numerical example.

4.1. Closed-loop system with VFO PID controller transient response numerical evaluation

Consider a closed-loop system with a plant described by difference Equation (12) with the following coefficients (Günther, 1986)

$$\begin{aligned} &A_2 = 1, A_1 = 1.9397, A_0 = 0.3804, B_2 = 0.0191, \\ &B_1 = -0.0666, B_0 = 0.0475, n_2 = m_2 = 2, \\ &n_1 = m_1 = 1, n_0 = m_0 = 0 \end{aligned}\,. \tag{47}$$

The controller is described by a VFODE of the form

$$v_k = K_P e_k + K_{I\,0}\Delta_k^{(m_{1,k})}e_k + K_{D\,0}\Delta_k^{(m_{2,k})}e_k\,. \tag{48}$$

We assume that the following constraint

$$-20 \le |v_k| \le 20\,, \tag{49}$$

is imposed on the controlling signal.

To preserve the maximum value of the controlling signal for $k = 0$, the coefficients K_p, K_I, K_D must satisfy the equality $K_p + K_I + K_D = 20$ The controller gains chosen are $K_p = 16{,}375$, $K_I = 3{,}125$ and $K_D = 0{,}5$ and the orders $m_{1,0} = -1$ and $m_{2,0} = 1$ can be chosen freely. The VFOs are selected and plotted in Fig.4

$$m_{1,k} = \begin{cases} -1\,(\text{arbitrary}) & \text{for} \quad k = 0 \\ -1.2505 & \text{for} \quad k = 1 \\ -1 + 0.8e^{-(k-1)} & \text{for} \quad 2 \le k \le 9 \\ -1 & \text{for} \quad k \ge 10 \end{cases}, \tag{50}$$

$$m_{2,k} = \begin{cases} 1\,(\text{arbitrary}) & \text{for} \quad \mathrm{k} = 0 \\ 0.9596 & \text{for} \quad \mathrm{k} = 1 \\ 1 + 6.034e^{-0.05(k-1)} & \text{for} \quad 2 \le k \le 9 \\ 1 & \text{for} \quad k \ge 10 \end{cases}, \tag{51}$$

It should observed that the exponential order functions in formulae (50) and (51) are non-unique. There is a wide choice of other functions. It seems important to preserve the condition $m_{1,k} = -1$ and $m_{2,k} = 1$ for all $k \geq k_I$ (when the system achieves its steady-state). Over mentioned interval the closed-loop system can be described by classical (integer order) DE. This requirement eliminates real-time calculation problems.

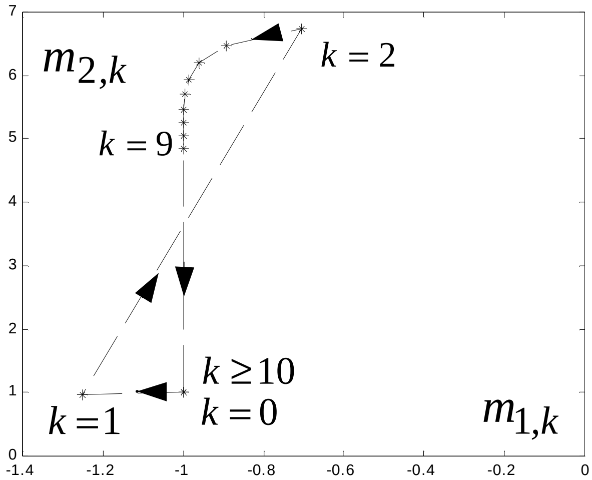

Fig. 4. VFOs $m_{1k} - m_{2k}$ trajectory

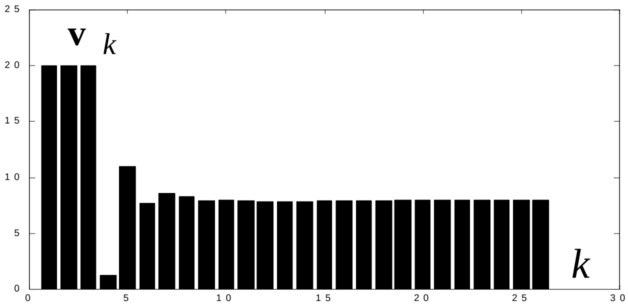

Fig. 5. The closed-loop system-controlling signal $\mathbf{v}_k$

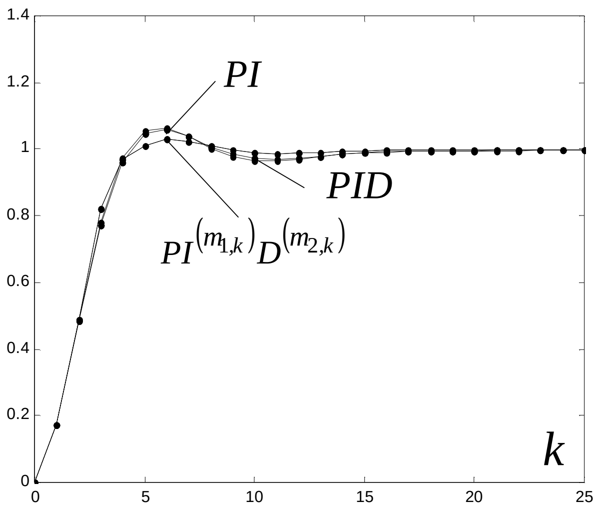

Fig. 6. The closed-loop system outputs with different controllers

In Fig. 3 the VFO satisfying Equations (50) and (51) are presented in orders plane $m_1 m_2$. It should be noted that after a finite number of time instants, the VFODE based control algorithm becomes a simple integer-order one, and as a result, a linear increase of samples considered in a calculation process (growing "calculation tail" or "finite memory problem") can be avoided. Thus, it is not necessary to simplify an algorithm by cutting the "calculation tail".

The control algorithm has PID controller properties (Günther, 1986; Ifeachor and Jervis, 1993; Isermann, 1988; Ogata, 1987). To avoid a growing number of samples (so called "calculation tail" (Podlubny, 1999)), after a finite number of a control algorithm steps the orders m_{1k} and m_{2k} become integers. Thus, the "calculation tail" has automatically been cut off. This should be achieved in a quasi-steady-state of the closed-loop system response. We assume zero initial conditions of the plant and the controller. In Fig. 5 we show the controlling signal $\mathbf{v}_k$.

Fig. 6 presents the plant output $\mathbf{y}_k(.)$ for the case when $\mathbf{d}_k = \mathbf{d}_{ik} = \mathbf{n}_k = 0$ and the reference signal is the unit discrete-time step function $\mathbf{r}_k = \mathbf{1}_k = [1 \;\; 1 \;\; 1 \;\; \cdots \;\; 1]^{\mathrm{T}}_{(k+1)\times 1}$. In the same Figure we also show the system responses with classical PI and PID controllers (Günther, 1986; Ifeachor and Jervis, 1993; Isermann, 1988) described by the discrete transfer functions

$$R_{PI}(z) = \frac{r_0 + r_1 z^{-1}}{1 + p_1 z^{-1}}, \tag{52}$$

where: $r_0 = 20$, $r_1 = -17$, $p_1 = -1$ and

$$R_{PID}(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 + p_1 z^{-1}}, \tag{53}$$

where: $q_0 = 20$, $q_1 = -16.572$, $q_2 = -0.4337$, $p_1 = -1$.

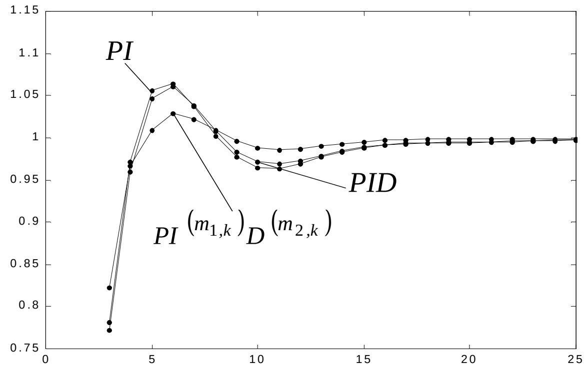

Fig. 7. The closed-loop system outputs with different controllers restricted to k ∈ [3 25]

The PI and PID controller parameters were obtained to preserve the minimum value of the performance criterion on the assumption of a bounded controlling signal (49).

$$I_{k_{\min}, k_{\max}} = \min_{|u_k| \leq 20} \sum_{i=k_{\min}}^{k_{\max}} e_k^2, \; k_{\min} = 0, \; k_{\max} = 50, \tag{54}$$

Assumption (49), required by practical applications, is so strong that for PI, PID and proposed $\mathrm{PI}^{(m_{1,k})}\mathrm{D}^{(m_{2,k})}$ control strategy, $v_0 = v_1 = 20$. This causes that the closed-loop system responses for $k = 0, 1, 2$ to be the same for all strategies. Owing to this, the PI and PID controllers produce very similar responses.

In Fig. 7 the same responses are presented over the time interval $k \in [3,\ 25]$.

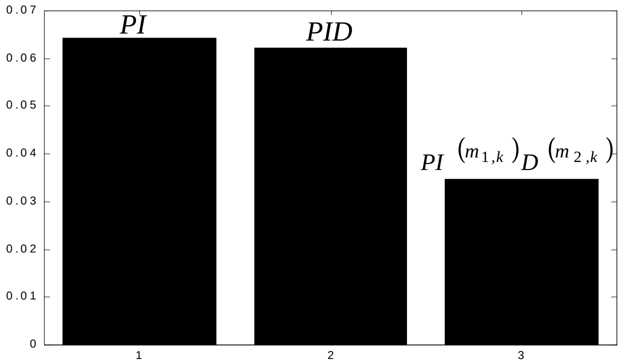

Fig. 8. $I_{PI^{(m1,k)}D^{(m2,k)}}$, $I_{PID,3,50}$, $I_{PI^{(m1,k)}D^{(m2,k)}} I_{PI,3,50}$ values performance criteria

The differences between $\mathbf{y}_{PI,k}$, $\mathbf{y}_{PID,k}$ and $\mathbf{y}_{PI^{(m1,k)}D^{(m2,k)},k}$ become significant when the following performance criteria are evaluated

$$I_{PI^{(m1,k)}D^{(m2,k}} = \min_{|u_k|\le 20} \sum_{i=3}^{50} e^2_{PI^{(m1,k)}D^{(m2,k)},k}$$

$$I_{PID,3,50} \min_{|u_k|\le 20} \sum_{i=3}^{50} e^2_{PID,k} \quad , \qquad (55)$$

$$I_{PI^{(m1,k)}D^{(m2,k}} I_{PI,3,50} = \min_{|u_k|\le 20} \sum_{i=3}^{50} e^2_{PI,k}$$

Its values are presented in Fig. 8.

5. FINAL CONCLUSIONS

The notion of the linear, time-invariant (with respect to coefficients), VFOs difference equations is applied to the discrete-time closed-loop system synthesis. A new description of linear time-invariant fractional-order closed-loop dynamical systems is investigated. As a practical application, a simple control strategy has been applied to a linear plant. It is non-unique. It appears that a large variety of advanced control strategies may effectively be applied in a real-time control. An open problem is how to design the VFOs depending on the closed-loop system error $m_{1,k}(\mathbf{e}_k)$ and $m_{2,k}(\mathbf{e}_k)$. An appropriate choice of order functions seems to be a fruitful task in further investigations in the case of plant parameters variations or uncertainties.

It is important to point out that applying fractional-order control strategies a linearly growing number of samples should be taken into calculations (linearly growing "calculation tail"). This can be avoided by introducing an assumption that for a quasi steady-state, the control strategy is described by an integer-orders difference equation.

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