NUMERICAL EVALUATION OF FRACTIONAL DIFFER-INTEGRAL OF SOME ELEMENTARY FUNCTIONS VIA INVERSE TRANSFORMATION

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Abstract: This paper presents methods of calculating fractional differ-integrals numerically. We discuss extensively the pros and cons of applying the Riemann-Liouville formula, as well as direct approach in form of The Grünwald-Letnikov method. We take closer look at the singularity, which appears when using classical form of Riemann-Liouville formula. To calculate Riemann-Liouville differ-integral we use very well-known techniques like The Newton-Cotes Midpoint Rule. We also use two Gauss formulas. By implementing transformation of the core integrand of Riemann-Liouville formula (we called it "the inverse transformation"), we would like to point the possible way of reducing errors when calculating it. The core of this paper is the subject of reducing the absolute error when calculating Riemann-Liouville differ-integrals of some elementary functions; we use our own C++ programs to calculate them and compare obtained results of all methods with, where possible, exact values, where not – with values obtained using excellent method of integration incorporated in Mathematica. We will not discuss complexity of numerical calculations. We will focus solely on minimization of the absolute errors.

1. INTRODUCTION

Fractional calculus is playing recently a major role in many scientific areas. The fractional-order derivative (FOD) or integral (FOI) are natural extensions of the well-known derivatives and integrals. This generalisation enables better physical phenomena identification (Oustaloup et al., 2005; Sabatier et al., 2007), analysis (Carpinteri and Mainardi, 1997; Chen et al., 2004; Kilbas et al., 2006; Michalski, 1993; Miller and Ross, 1993; Nishimoto, 1984, 1989, 1991, 1996; Oldham and Spanier, 1974; Oustaloup, 1995; Samko et al., 1993) and control (Machado, 2001, Ostalczyk, 2000, 2003a, b; Oustaloup, 1984). But there are still problems in numerical evaluation of the fractional-order derivatives or integrals (Deng, 2007; Diethelm, 1997; Gorenflo, 2001; Lubich, 1986; Mayoral, 2006; Podlubny, 1999; Schmidt and Amsler, 1999; Tuan and Gorenflo, 1995). In this paper several numerical methods applied to FOD/FOI calculation are compared, due to its accuracy. Appropriate conclusions and remarks are derived.

The paper is organised as follows. Firstly basic definitions of FOD and FOI are given. In Section 3 short review of numerical methods used in calculation of the improper integrals is given. Section 4 presents functions subjected to the fractional differentiation and integration. In Section 5 main results are presented. Finally, the conclusions are given.

2. MATHEMATICAL PRELIMINARIES

There are several formulas, which can be used to calculate differ-integrals numerically. One of them is Grünwald-Letnikov and second one Riemann-Liouville, formula (Ostalczyk, 2000; Podlubny, 1999; Samko et al., 1993). They distinct from each other in one main way: Grünwald-Letnikov formula derives from differential quotient and Riemann-Liouville from multiple integrals.

This paper shows the pros and cons of applying the Riemann-Liouville formula. Also, the ideas how to reduce absolute errors when calculating it numerically. The Grünwald-Letnikov formula is used for comparing purposes only. The accuracy reached by this method as reference.

2.1. The Grünwald-Letnikov formula of a fractional-order differ-integral (GrLet)

The derivative of a real order $v > 0$ (for the integral we use order $-v < 0$) of a continuous bounded function $f(t)$ is defined as follows

$$ {}_{t_0}D_t^{\nu} f(t) = \lim_{\substack{h\to 0\\ t-t_0=kh}} \frac{\sum_{i=0}^{\frac{t-t_0}{h}} a_i^{(\nu)} f(t-hi)}{h^{\nu}} \tag{1}$$

where

$$ a_i^{(\nu)} = \begin{cases} 1 & \text{for} \quad i=0 \\ a_{i-1}^{(\nu)}\left(1-\frac{1+\nu}{i}\right) & \text{for} \quad i=1,2,3,\ldots \end{cases} \tag{2}$$

2.2. The Riemann-Liouville formula of a fractional-order differ-integral (RL)

The definite Riemann-Liouville integral of the real function $f(t)$ of the $v > 0$ order is defined as follows:

$$ {}_{t_0}I_t^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{t_0}^{t} (t-\tau)^{\nu-1} f(\tau) d\tau. \quad (3)$$

where: $t_0, t -$ integration range, which comply with the condition $-\infty < t_0 < t < \infty$, $\Gamma(v) -$ Euler's Gamma function.

Before we define the Riemann-Liouville derivative, we have to describe natural number n, which comply with the condition:

$$ n = [\nu] + 1. \quad (4)$$

n also denotes an order of classical derivative.

The Riemann-Liouville derivative of the real function $f(t)$ of the $v > 0$ order is defined as follows:

$$ {}_{t_0}D_t^{\nu} f(t) = \sum_{i=0}^{n-1} \frac{(t-t_0)^{i-\nu} f^{(i)}(t_0)}{\Gamma(i+1-\nu)} + + \frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} (t-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau \quad (5)$$

3. SHORT REVIEW OF FUNDAMENTAL METHODS OF NUMERICAL INTEGRATION AND TESTED FUNCTIONS

In the process of calculating differ-integrals it is necessary to calculate a value of the definite integral over the range $[t_0, t]$. Usually it is interpolated with the following formula

$$ \int_{t_0}^{t} f(t) dt = \sum_{k=0}^{L} A_k f(t_k) + R. \quad (6)$$

The right side of the equation is called quadrature, in which $t_k -$ denotes quadrature nodes, $A_k -$ quadrature coefficients (weights), $L -$ number of intervals in interpolation and $R -$ the remainder.

The above formula is shared by all quadratures. The difference lies in the algorithms of calculating their nodes and coefficients.

We used following formulas to calculate differ-integrals:

- Riemann-Liouville differ-integral (RL);
- Modified Riemann-Liouville differ-integral via mentioned at the beginning – inverse transformation (mRL).

Additionally we use Grünwald-Letnikov differ-integral formula (GrLET).

Our C++ programs which were developed especially for the purpose of this experiment used following methods of numerical integration while applying formulas (RL, mRL):

- Newton-Cotes quadrature, Midpoint Rule (NCM);
- Gauss-Legendre quadrature (GaLEG);
- Gauss-Laguerre quadrature (GaLAG).

We have chosen three basic functions

$$ f(t) = t^p 1(t), t \in (0;1) \text{ for } p = 0,1,2 \quad (7)$$

where

$$ 1(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases} \quad (8)$$

is the Heaviside function. For functions (7) we calculate two types of expressions: ${}_{t_0}D_t^{\nu} f(t)$ and ${}_{t_0}I_t^{\nu} f(t)$. In Tab. 1 different methods specifications are collected.

Tab. 1. Important parameters used in integration rules

Method /weight function	h / A_k	t_k	$R \leq$
NCM	$h = \frac{t-t_0}{L}$	$t_k = t_0 + (k + 1/2)h$	$\frac{h^3}{24} \lvert f^{(II)}(\zeta) \rvert$, $\zeta \in [t_0, t]$
GaLEG $p(x) = 1$	$A_k = \frac{2}{(1-t_k^2)[P_n'(t_k)]^2}$	Abscissas of the Legendre polynomial $P_n(x)$ of desired grade x_k. $t_k = \frac{t-t_0}{2} x_k + \frac{t-t_0}{2}$	$\frac{t-t_0}{2016000} f^{(VI)}(\zeta)$, $\zeta \in [t_0, t]$
GaLAG $p(x) = e^{-x}$	$A_k = \frac{(n!)^2}{x_k [L_n'(x_k)]^2}$	Abscissas of the Laguerre polynomial $L_n(x)$ of desired grade x_k.	$\frac{(n!)^2}{(2n)!} f^{(2n)}(\zeta)$ $\zeta \in \langle 0; +\infty)$

Our goal was to figure out how the methods will perform when using the smallest, arbitral chosen, number of sample points:

For the method GrLET and NCM we used L=4,8,16,24,32,100,300 and 600 intervals.

For both Gauss methods – L=4,8,16,24 and 32 intervals only.

It is widely known, that number of L greater than 30-40 for the Gauss methods often causes the error rise rapidly. Sometimes 100% and more! That's why you will encounter empty fields in all tables with results for these methods.

4. THE INVERSE TRANSFORMATION (mRL) EXPLAINED

As we remember the Riemann-Liouville differ-integral formula includes improper integral which has singularity at end of the integration range. For example for t=1: $\int_0^1 (1-x)^{v-1} f(x) dx$. The variable changes $1 - x = 1/t^{\alpha}$, $\alpha = 1, 2, 3, \ldots$ and $t - 1 = u$ convert the improper integral into one, that, after extracting weight function $p(x) = e^{-x}$ can then be calculated by the Gauss-Laguerre quadrature formula $\int_0^{\infty} e^{-x} f(t) dx$, which were developed to deal with such problems.

Yet more: with the parameter α we can control the convergence of the integrand, which plays major role when obtaining best results while the order of differ-integral changes. As you will notice further, there exists very close relation between order of differ-integral and the value of parameter α. We will use it to our advantage.

5. THE TEST RESULTS

First ${}_{t_0}I_t^{\nu} f(t)$ of function $f(t) = t^0 1(t) = 1(t)$, for $t \in (1,1)$, $\nu = 0.2, 0.5, 0.8$ using modified Riemann-Liouville differ-integral formula via mRL

$${}_{t_0}I_t^{\nu} f(t) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^t \frac{\alpha}{\left(\frac{1}{(1+t)^{\alpha}}\right)^{\nu-1}(1+t)^{\alpha+1}} dt \qquad (9)$$

was evaluated. The results are presented in Tab. 2a – 2c. Related absolute errors are plotted in Figs. 1a – 1c

Tab. 2a. Obtained values of absolute error for $\nu = 0.2$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	5.427e-01	4.587e-01	2.206e-02	4.822e-02
8	4.725e-01	3.557e-01	1.097e-02	4.529e-03
16	4.114e-01	2.729e-01	5.465e-03	1.297e-04
24	3.794e-01	2.330e-01	3.639e-03	1.417e-05
32	3.582e-01	**2.081e-01**	2.728e-03	**1.076e-10**
100	2.852e-01	-	8.718e-03	-
300	2.289e-01	-	2.905e-04	-
600	**1.993e-01**	-	**1.452e-04**	-

Tab. 2b. Obtained values of absolute error for $\nu = 0.5$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	1.699e-01	1.039e-01	3.463e-02	6.951e-02
8	1.205e-01	5.781e-02	1.748e-02	7.111e-03
16	8.527e-02	2.977e-02	8.780e-03	1.909e-04
24	6.964e-02	2.005e-02	5.861e-03	1.096e-05
32	6.032e-02	**1.511e-02**	4.399e-03	**9.725e-10**
100	3.413e-02	-	1.410e-03	-
300	1.970e-02	-	4.701e-04	-
600	**1.393e02**	-	**2.351e-04**	-

Tab. 2c. Obtained values of absolute error for ν=0.8

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	3.048e-02	1.439e-02	2.070e-02	6.501e-02
8	1.765e-02	5.183e-03	1.055e-02	6.162e-03
16	1.017e-02	1.792e-03	5.321e-03	1.771e-04
24	7.362e-03	9.516e-04	3.558e-03	1.033e-05
32	5.852e-03	**6.054e-04**	2.672e-03	**2.545e-10**
100	2.354e-03	-	8.577e-03	-
300	9.777e-04	-	2.862e-04	-
600	**5.615e-04**	-	**1.431e-04**	-

In Tab. 3 optimal values of α as functions of orders are presented. Convergence of modified integrands – Fig. 2.

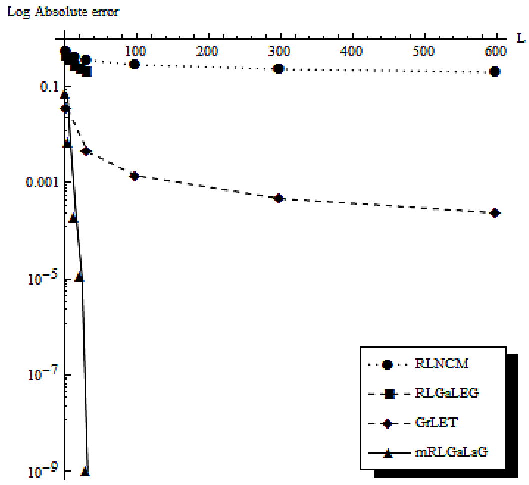

Fig. 1a. Values of absolute error for $\nu = 0.2$

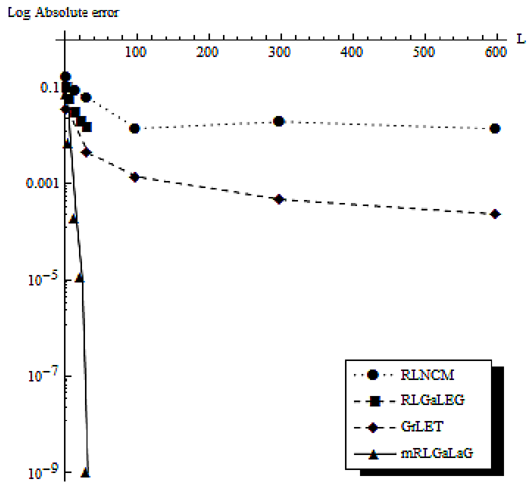

Fig. 1b. Values of absolute error for $\nu = 0.5$

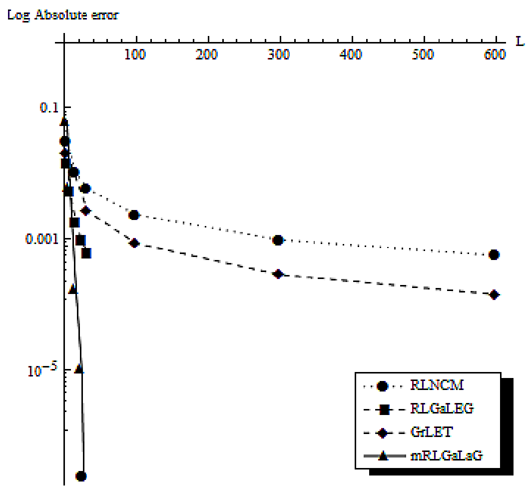

Fig. 1c. Values of absolute error for ν=0.8

Tab. 3. Lowest values of absolute error obtained for optimal values of α depending on ν (mRL GaLAG)

α	$\nu = 0.2$	$\nu = 0.5$	ν=0.8
12.95	**1.076e-08**	4.591e-05	6.842e-04
5.97	3.171e-03	**9.725e-10**	7.705e-06
3.71	2.975e-02	1.020e-04	**2.545e-09**

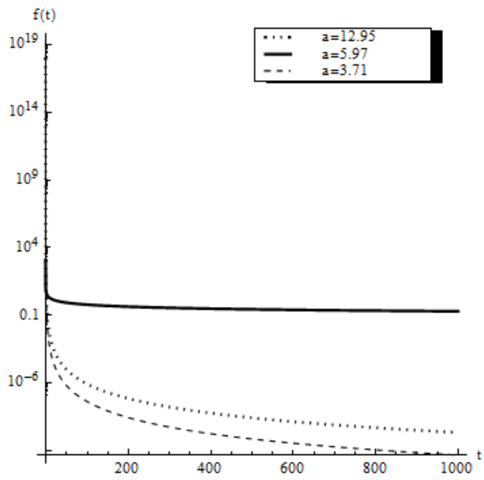

Fig. 2. Convergence of integrand (9) for optimal values of α depending on ν (mRL GaLAG)

Next similar integrals are obtained for function $f(t) = t^1 1(t)$ for $t \in (0,1)$, $\nu = 0.2, 0.5, 0.8$. This time a modified Riemann-Liouville differ-integral formula via mRL assumes the form

$$ {}_{t_0}I_t^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^t \frac{\alpha\left(1-\frac{1}{(1+t)\alpha}\right)}{\left(\frac{1}{(1+t)^{\alpha}}\right)^{\nu-1} (1+t)^{\alpha+1}} dt \qquad (10) $$

The results are presented in Tabs. 4a – 4c and related plots are included in Figs. 3a – 3c.

Tab. 4a. Obtained values of absolute error for $\nu = 0.2$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	5.442e-01	4.592e-01	2.608e-02	6.521e-02
8	7.339e-01	3.558e-01	1.332e-02	1.380e-02
16	4.118e-01	2.729e-01	6.734e-02	6.505e-04
24	3.796e-01	2.330e-01	4.505e-02	2.389e-05
32	3.583e-01	**2.081e-01**	3.385e-03	**9.019e-07**
100	2.852e-01	-	1.087e-03	-
300	2.289e-01	-	3.628e-04	-
600	**1.993e-01**	-	**1.815e-04**	-

Tab. 4b. Obtained values of absolute error for $\nu = 0.5$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	1.735e-01	6.806e-02	6.806e-02	8.860e-02
8	1.218e-01	5.790e-02	3.463e-02	2.299e-03
16	8.576e-01	2.972e-02	1.747e-02	1.406e-03
24	6.991e-01	2.005e-02	1.168e-02	7.497e-04
32	6.050e-01	**1.512e-02**	8.775e-03	**8.816e-07**
100	3.416e-01	-	2.816e-03	-
300	1.971e-01	-	9.399e-04	-
600	**1.393e-01**	-	**4.700e-04**	-

Tab. 4c. Obtained values of absolute error for $\nu = 0.8$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	3.237e-02	1.459e-02	1.055e-01	9.402e-02
8	1.826e-02	5.203e-03	5.320e-02	4.341e-02
16	1.036e-02	1.793e-03	1.036e-02	5.540e-03
24	7.459e-03	9.521e-04	1.784e-02	5.880e-04
32	5.911e-03	**6.056e-04**	1.339e-02	**1.202e-06**
100	2.362e-03	-	4.292e-03	-
300	9.789e-04	-	1.431e-03	-
600	**5.619e-04**	-	**7.157e-04**	-

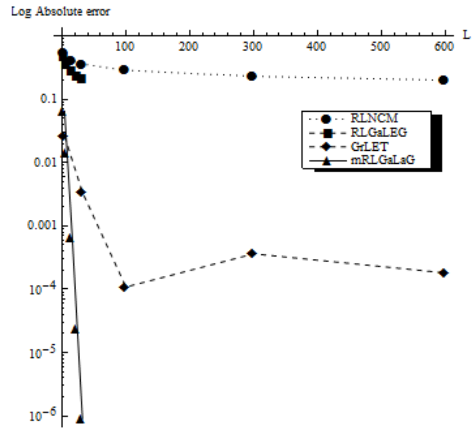

Fig. 3a. Values of absolute error for $\nu = 0.2$

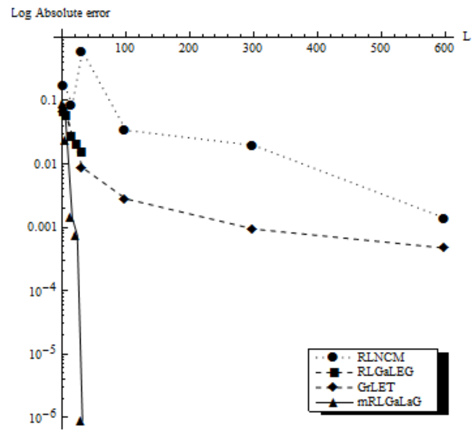

Fig. 3b. Values of absolute error for $\nu = 0.5$

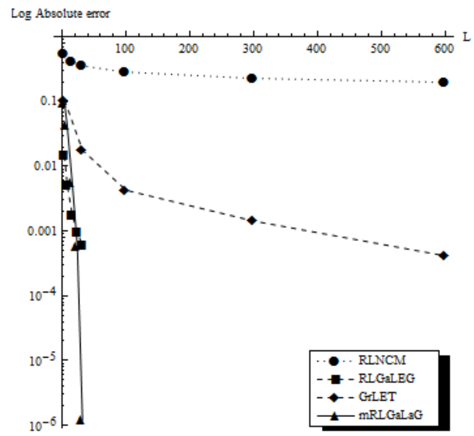

Fig. 3c. Values of absolute error for $\nu = 0.8$

In Tab. 5 optimal values of α as functions of orders are presented. Convergence of modified integrands – Fig. 4.

Tab. 5. Lowest values of absolute error obtained for optimal values of α depending on ν (mRL GaLAG)

α	$\nu = 0.2$	$\nu = 0.5$	$\nu = 0.8$
9.090	**1.202e-06**	1.112e-03	3.591e-03
4.341	1.604e-02	**8.816e-07**	5.725e-05
2.900	6.567e-02	8.919e-04	**9.019e-07**

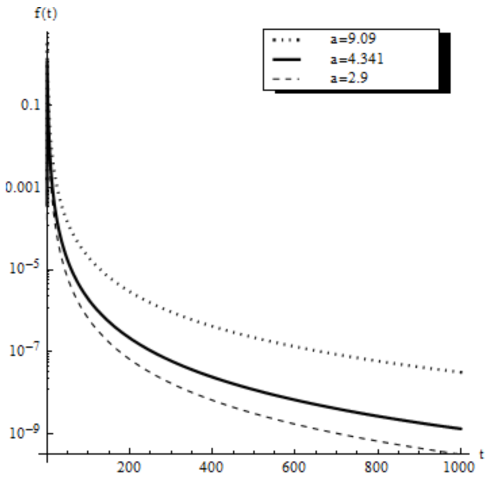

Fig. 4. Convergence of integrand (10) for optimal values of α depending on ν (mRL GaLAG)

Tab. 6a. Obtained values of absolute error for $\nu = 0.2$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	5.463e-01	4.579e-01	4.493e-02	8.935e-02
8	4.742e-01	5.222e-01	2.258e-02	5.645e-03
16	4.121e-01	1.795e-01	1.132e-02	9.926e-03
24	3.798e-01	9.529e-01	7.551e-03	3.905e-03
32	3.585e-01	**6.057e-01**	1.088e-03	**1.330e-03**
100	2.853e-01	-	1.814e-03	-
300	2.290e-01	-	6.050e-04	-
600	**1.993e-01**	-	**3.025e-04**	-

Tab. 6b. Obtained values of absolute error for $\nu = 0.5$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	1.789e-01	1.106e-01	9.550e-02	9.566e-02
8	1.236e-01	5.800e-02	4.740e-02	3.233e-03
16	8.638e-01	2.980e-02	2.360e-02	6.917e-03
24	7.024e-01	2.006e-02	1.571e-02	1.623e-03
32	6.071e-01	**1.512e-02**	1.178e-02	**3.450e-04**
100	3.420e-01	-	3.764e-03	-
300	1.972e-01	-	1.254e-03	-
600	**1.394e-01**	-	**6.269e-04**	-

Finally we calculate ${}_{t_0}I_t^{\nu} f(t)$ of function $f(t) = t^2 1(t)$ for $t \in (0,1)$, $\nu = 0.2, 0.5, 0.8$. The modified Riemann-Liouville differ-integral formula via mRL assumes the form

$$ {}_{t_0}I_t^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^t \frac{\alpha\left(1-\frac{1}{(1+t)\alpha}\right)^2}{\left(\frac{1}{(1+t)^{\alpha}}\right)^{\nu-1}(1+t)^{\alpha+1}} dt \qquad (11) $$

The results are presented in Tab. 6a – 6c and related plots are included in Figs. 5a – 5c. In Tab. 7 optimal values of α as functions of orders are presented. Convergence of modified integrands – Fig. 10.

Tab. 6c. Obtained values of absolute error for $\nu = 0.8$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	3.819e-02	1.480e-02	1.255e-01	2.393e-02
8	1.986e-02	5.222e-03	6.121e-02	8.463e-03
16	1.081e-02	1.795e-03	3.021e-02	1.875e-03
24	7.667e-03	9.529e-04	2.006e-02	2.769e-04
32	6.033e-03	**6.057e-04**	1.501e-02	**4.059e-05**
100	2.377e-03	-	4.782e-03	-
300	9.808e-04	-	1.592e-03	-
600	**5.624e-04**	-	**7.956e-04**	-

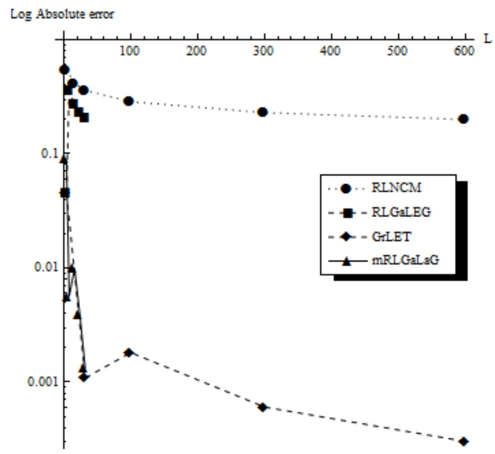

Fig. 5a. Values of absolute error for $\nu = 0.2$

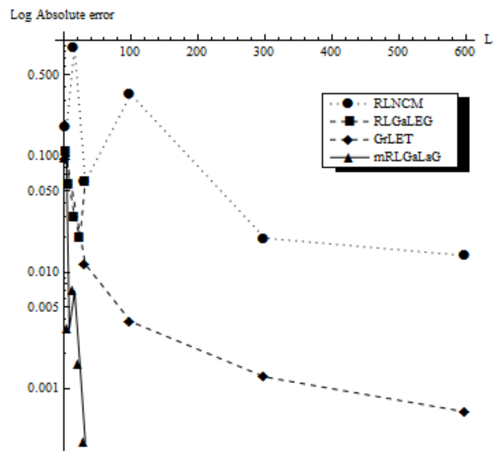

Fig. 5b. Values of absolute error for $\nu = 0.5$

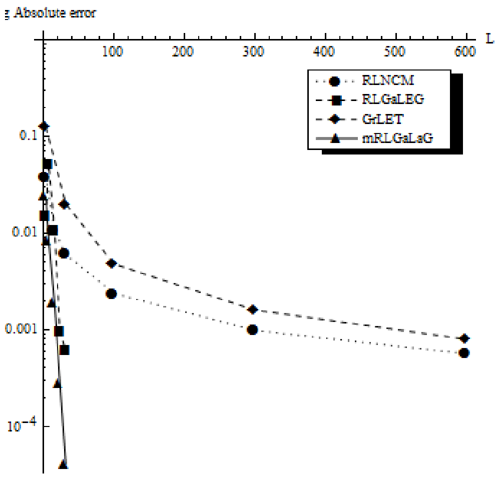

Fig. 5c. Values of absolute error for $v = 0.8$

Tab. 7. Lowest values of absolute error obtained for optimal values of α depending on v(mRL GaLAG)

α / ν	0.2	0.5	0.8
7.91	**1.330e-03**	3.238e-03	6.127e-03
5.05	8.027e-03	**3.640e-04**	8.272e-04
2.90	6.564e-02	9.081e-04	**4.059e-05**

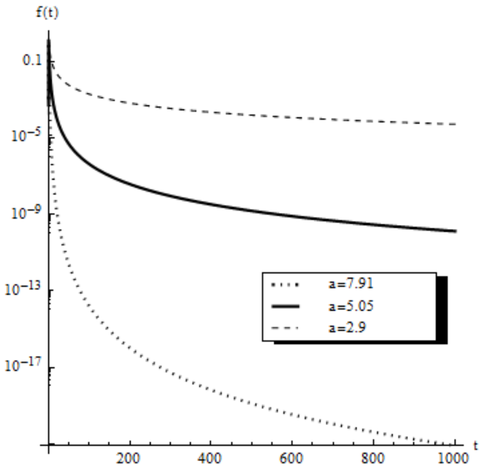

Fig. 6. Convergence of integrand (11) for optimal values of α depending on v (mRL GaLAG)

Now a problem of the fractional derivative ${}_{t_0}D_t^{v} f(t)$ of function $f(t) = t^0 1(t)$ for $t \in (0, 1)$, $v = 0.2, 0.5, 0.8$ is considered. We assume $f(0) = 1$ and calculate

$$[n] = \nu + 1 \tag{12}$$

Then the modified Riemann-Liouville differ-integral formula via mRL takes the form

$$ {}_{t_0}D_t^{\nu} f(t) = \tag{13}$$

$$ = \frac{f(0)}{\Gamma(1-\nu)} + \frac{1}{\Gamma(n-\nu)} \overbrace{\int_0^\infty e^t \frac{\alpha \cdot 0}{\left(\frac{1}{(1+t)^\alpha}\right)^{n-\nu-1} (1+t)^{\alpha+1}} dt}^{=0} $$

One can realize that the above value depends, in this case solely on accuracy of the inverse gamma function. The obtained results are presented in Tabs. 8a – 8c and related plots are included in Figs. 7a – 7c.

Tab. 8a. Obtained values of absolute error for $v = 0.2$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	**0.0**	**0.0**	2.777e-02	**0.0**
8	**0.0**	**0.0**	1.377e-02	**0.0**
16	**0.0**	**0.0**	6.562e-03	**0.0**
24	**0.0**	**0.0**	4.348e-03	**0.0**
32	**0.0**	**0.0**	3.251e-03	**0.0**
100	**0.0**	-	1.034e-03	-
300	**0.0**	-	3.439e-04	-
600	**0.0**	-	**1.718e-04**	-

Tab. 8b. Obtained values of absolute error for $v = 0.5$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	**0.0**	**0.0**	6.081e-02	**0.0**
8	**0.0**	**0.0**	2.829e-02	**0.0**
16	**0.0**	**0.0**	1.376e-02	**0.0**
24	**0.0**	**0.0**	9.011e-03	**0.0**
32	**0.0**	**0.0**	6.721e-03	**0.0**
100	**0.0**	-	2.127e-03	-
300	**0.0**	-	7.065e-04	-
600	**0.0**	-	**3.529e-04**	-

Tab. 8c. Obtained values of absolute error for $v = 0.8$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	**0.0**	**0.0**	4.894e-02	**0.0**
8	**0.0**	**0.0**	2.177e-02	**0.0**
16	**0.0**	**0.0**	1.031e-02	**0.0**
24	**0.0**	**0.0**	6.758e-03	**0.0**
32	**0.0**	**0.0**	5.026e-03	**0.0**
100	**0.0**	-	1.581e-03	-
300	**0.0**	-	5.242e-04	-
600	**0.0**	-	**2.617e-04**	-

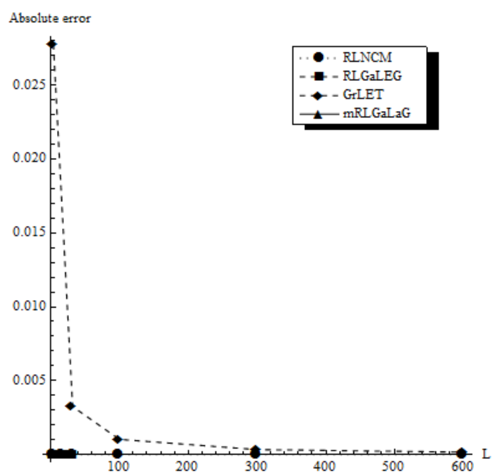

Fig. 7a. Values of absolute error for $v = 0.2$

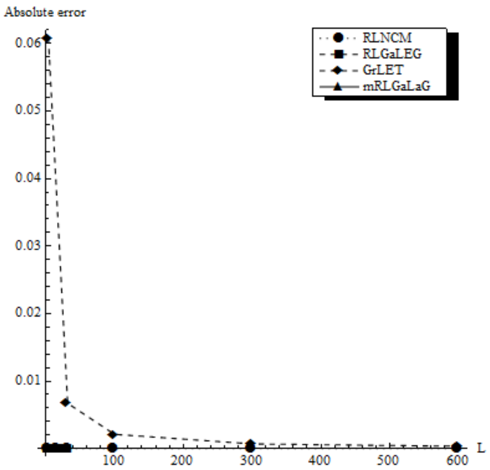

Fig. 7b. Values of absolute error for $v = 0.5$

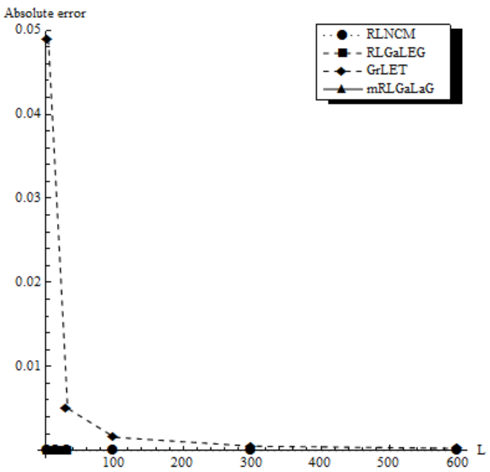

Fig. 7c. Values of absolute error for $v = 0.8$

Tab. 9a. Obtained values of absolute error for $v = 0.2$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	3.048e-02	1.439e-02	2.070e-02	6.501e-02
8	1.765e-02	5.183e-03	1.055e-02	6.616e-03
16	1.017e-02	1.792e-03	5.321e-03	1.771e-04
24	7.362e-03	9.516e-04	3.558e-03	1.033e-05
32	5.852e-03	**6.054e-04**	2.672e-03	**2.545e-09**
100	2.354e-03	-	8.577e-04	-
300	9.776e-04	-	2.862e-04	-
600	**5.615e-04**	-	**1.431e-04**	-

Tab. 9b. Obtained values of absolute error for $v = 0.5$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	1.699e-01	1.093e-02	3.463e-02	7.006e-02
8	1.205e-01	5.781e-02	1.748e-02	7.148e-03
16	8.527e-02	2.977e-02	8.780e-03	1.932e-04
24	6.964e-02	2.005e-02	5.861e-03	1.103e-05
32	6.032e-02	**1.511e-02**	4.399e-03	**3.353e-08**
100	3.413e-02	-	1.410e-03	-
300	1.970e-02	-	4.700e-04	-
600	**1.393e-02**	-	**2.350e-04**	-

Tab. 9c. Obtained values of absolute error for $v = 0.8$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	5.424e-01	4.587e-01	2.206e-02	4.781e-02
8	4.725e-01	3.557e-01	1.097e-02	4.483e-03
16	4.114e-01	2.729e-01	5.465e-03	1.293e-04
24	3.794e-01	2.330e-01	3.639e-03	1.452e-05
32	3.582e-01	**2.081e-01**	2.728e-03	**4.105e-08**
100	2.852e-01	-	8.718e-04	-
300	2.289e-01	-	2.905e-04	-
600	**1.993e-01**	-	**1.452e-04**	-

Next similar derivative is obtained for function $f(t) = t^1 1(t)$ for $t \in (0,1)$, $v = 0.2, 0.5, 0.8$. Under a condition (12) modified Riemann-Liouville differ-integral formula via mRL assumes the form

$$ {}_{t_0}D_t^{\nu} f(t) = \qquad (14)$$

$$ = \frac{1}{\Gamma(n-\nu)} \int_0^{\infty} e^t \frac{\alpha}{\left(\frac{1}{(1+t)^{\alpha}}\right)^{n-\nu-1} (1+t)^{\alpha+1}} dt $$

The obtained results are presented in Tabs. 9a – 9c and related plots are included in Figs. 8a – 8c. In table 10 optimal values of α as functions of orders are presented. Convergence of modified integrands – Fig. 9.

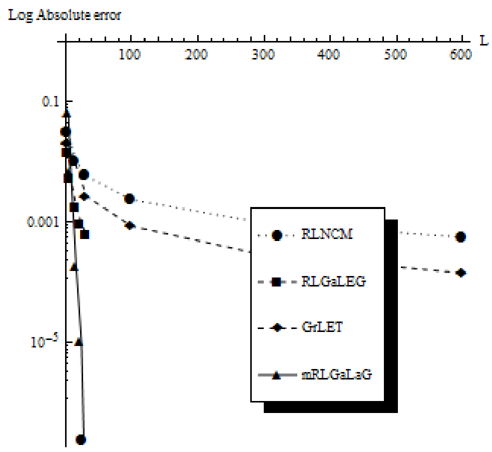

Fig. 8a. Values of absolute error for $v = 0.2$

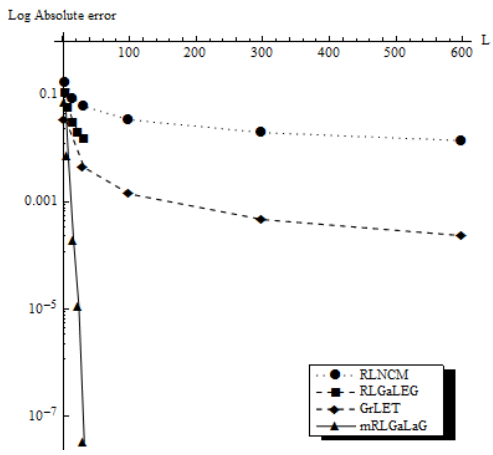

Fig. 8b. Values of absolute error for $v = 0.5$

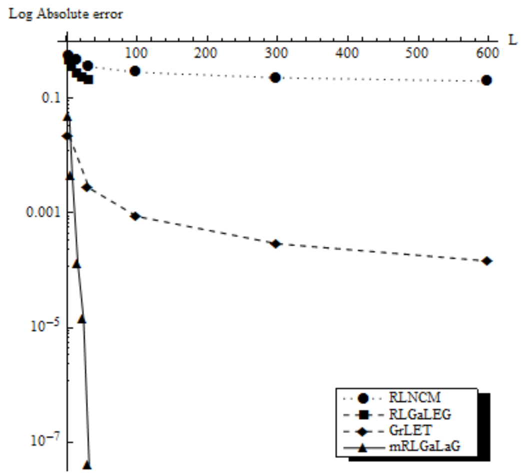

Fig. 8c. Values of absolute error for $\nu = 0.8$

Tab. 10. Lowest values of absolute error obtained for optimal values of α depending on ν (mRL GaLAG)

α	$\nu = 0.2$	$\nu = 0.5$	$\nu = 0.8$
12.90	6.691e-04	4.489e-05	**4.101e-08**
6.000	7.922e-06	**3.353e-08**	3.077e-03
3.710	**2.545e-09**	1.020e-04	2.975e-02

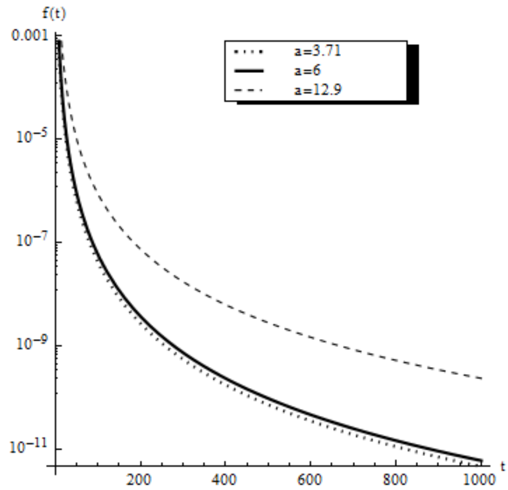

Fig. 9. Convergence of integrand (14) for optimal values of α depending on ν (mRL GaLAG)

Finally we calculate ${}_{t_0}D_t^{\nu} f(t)$ of function $f(t) = t^2 1(t)$ for $t \in (0,1)$, $\nu = 0.2, 0.5, 0.8$. Then under the condition (12) modified Riemann-Liouville differ-integral formula via mRL assumes the form

$${}_{t_0}D_t^{\nu} f(t) = \qquad (15)$$

$$= \frac{1}{\Gamma(n-\nu)} \int_0^{\infty} e^t \frac{2\alpha\left(1-\frac{1}{(1+t)\alpha}\right)}{\left(\frac{1}{(1+t)^{\alpha}}\right)^{n-\nu-1} (1+t)^{\alpha+1}} dt$$

The obtained results are presented in Tabs. 11a – 11c and related plots are included in Figs. 10a – 10c. In Tab. 12 optimal values of α as functions of orders are presented. Convergence of modified integrands – Fig. 11.

Tab. 11a. Obtained values of absolute error for $\nu = 0.2$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	6.473e-02	2.918e-02	5.225e-02	1.804e-01
8	3.651e-02	1.041e-02	2.648e-02	2.759e-02
16	2.073e-02	3.587e-03	1.133e-02	1.031e-03
24	1.492e-02	1.094e-03	8.907e-03	4.778e-05
32	1.182e-02	**1.211e-03**	6.688e-03	**1.844e-06**
100	4.724e-03	-	2.145e-03	-
300	1.958e-03	-	7.155e-04	-
600	**1.124e-03**	-	**3.578e-04**	-

Tab. 11b. Obtained values of absolute error for $\nu = 0.5$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	3.469e-01	2.198e-01	1.373e-01	1.772e-01
8	2.436e-01	1.158e-02	6.960e-02	4.600e-02
16	1.715e-01	5.957e-02	3.503e-02	2.813e-03
24	1.398e-01	4.011e-02	2.341e-02	1.502e-04
32	1.210e-01	**3.023e-02**	1.757e-02	**1.736e-06**
100	6.832e-02	-	5.363e-03	-
300	3.942e-02	-	1.880e-03	-
600	**2.787e-02**	-	**9.402e-04**	-

Tab. 11c. Obtained values of absolute error for $\nu = 0.8$

L	RL NCM	RL GaLEG	GrLET	mRL GaLAG
4	1.880e-00	9.183e-01	2.146e-01	1.833e-01
8	9.466e-01	7.117e-01	1.081e-01	8.234e-02
16	8.235e-01	5.458e-01	5.426e-02	1.002e-02
24	7.592e-01	4.659e-01	3.622e-02	9.573e-04
32	7.166e-01	**4.161e-01**	2.718e-02	**1.573e-06**
100	5.704e-01	-	8.708e-03	-
300	4.579e-01	-	2.904e-03	-
600	**3.968e-01**	-	**1.453e-03**	-

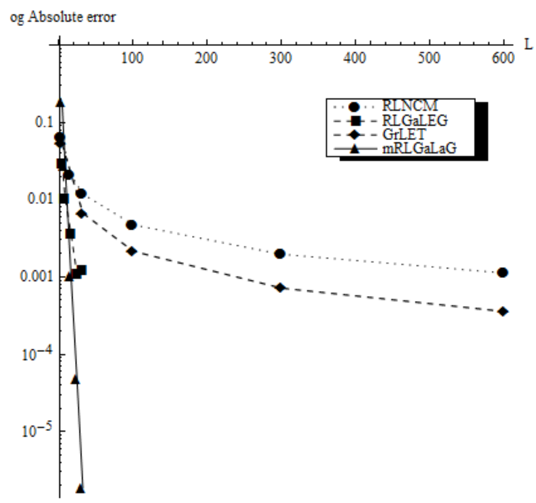

Fig. 10a. Values of absolute error for $\nu = 0.2$

Tab. 12. Lowest values of absolute error obtained for optimal values of α depending on v (mRL GaLAG)

α	$v = 0.2$	$v = 0.5$	$v = 0.8$
8.905	6.643e-03	2.044e-03	**1.573e-06**
4.341	1.145e-04	**1.763e-06**	3.201e-02
2.900	**1.804e-06**	1.784e-03	1.313e-01

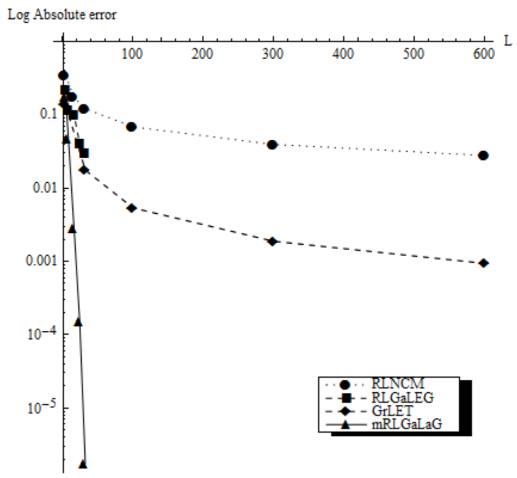

Fig. 10b. Values of absolute error for $v = 0.5$

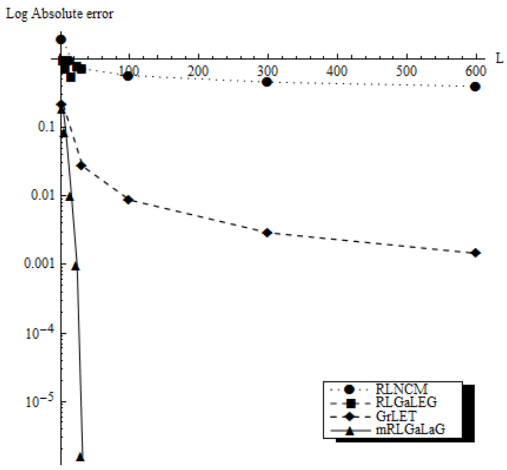

Fig. 10c. Values of absolute error for $v = 0.8$

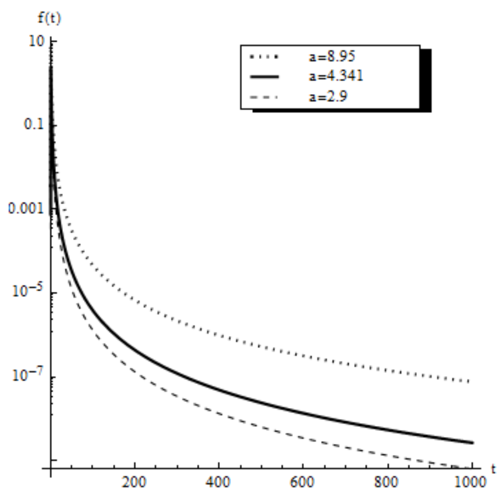

Fig. 11. Convergence of integrand (15) for optimal values of α depending on v (mRL GaLAG)

6. FINAL CONCLUSIONS

The results presented in Section 5 enable us to formulate the following conclusions:

1. The shape of the integrand does not influence accuracy of the calculations when using Grünwald-Letnikov method. The number of coefficients used – does. Using maximum number of 600 of coefficients we were able to obtain values with maximum $10e - 04$ accuracy.
2. The shape of integrand does influence accuracy of the calculations when applying advanced methods of integration to calculate differ-integrals using Riemann-Liouville formula.
3. The values of the integrand obtained using "pure" Riemann-Liouville formula are charged with great absolute error. This makes the formula often unsuitable in practical, technical applications.
4. This level of errors appeared because off the fact that the "core" integrand of the formula has "fast-changing" character and singularity at the end point of the integration range.
5. Applying inverse transformation of the integrand to "smash" the singularity allowed not only obtain much better results than by using Grünwald-Letnikov method, but often using radical reduced number of sampling points. This lowers the level of calculation complexity.
6. Applied transformation of variables and special substitute expression mentioned earlier to the "core" integrand allowed to lower the values of absolute errors about 2-6 times.
7. The values of absolute errors increased proportionally to the order of "complexity" (parameter p) of the function tested: for increasing values of p, absolute errors also increased proportionally.
8. Heaviside function – due to its character is the "domain" of Grünwald-Letnikov formula, but as the "complexity" of the function (other two functions tested) rises, if the integrand is modified, Newton-Cotes and Gauss-Laguerre rules seems to be appropriate to apply.
9. The Newton-Cotes Midpoint Rule is universal tool. Not only it does not depend so strongly as the Gauss-Legendre rule, on shape and changeability of the integrand, but also can be applied to integrands which have singularities at the both and/or end of the integration range.
10. Gauss-Laguerre rule, when applied to transformed integrand, seems to be the better way, not only because of the low values of absolute error, but also because of the fact, that these low values are obtained with only 5% sample points used by Grünwald-Letnikov method and Newton-Cotes Rule. This can dramatically reduce the complexity of the calculations.
11. Manipulation of the α variable in the inverse transformation allows to speed up the convergence of the integrand and lower the absolute error (notice figures: Convergence of integrand for optimal values of α depending on v (mRL GaLAG)). We noticed close relation between the values of α and v, when minimising the absolute error in calculations: for integrals – α should be reduced when v increases; for the derivatives – the other way round.

12. The logic of our programs needed only the degree of the desired polynomial as a input data. All other data were calculated "on the fly" (the polynomial itself, its derivative, abscissas and weights). In practical applications we can and should use tabulated values of abscissas and weights which were the subject of standardization all over the world. This can reduce more the complexity of calculations which then can make the method become yet more suitable in practical applications.

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