NECESSARY AND SUFFICIENT STABILITY CONDITIONS OF FRACTIONAL POSITIVE CONTINUOUS-TIME LINEAR SYSTEMS

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Abstract: Necessary and sufficient conditions for the asymptotic stability of fractional positive continuous-time linear systems are established. It is shown that the matrix A of the stable fractional positive system has not eigenvalues in the part of stability region located in the right half of the complex plane.

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2002). Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Simple conditions for practical stability of discrete-time linear systems have been proposed by Busłowicz and Kaczorek (2009) and next have been extended to robust stability of fractional discrete-time linear systems in Busłowicz (2010). The stability and stabilization of positive fractional linear systems by state-feedbacks have been analyzed in Kaczorek (2010, 2011b). The Hurwitz stability of Metzler matrices has been investigated in Narendra and Shorten (2010) and some new tests for checking the asymptotic stability of positive standard and fractional linear systems have been proposed in Kaczorek (2011a).

In this paper necessary and sufficient conditions for the asymptotic stability of fractional positive continuous-time linear systems will be established. It will be shown that the matrix A of the stable fractional positive system has not eigenvalues in the part of stability region located in the right half of the complex plane.

The paper is organized as follows. In section 2 basic definitions and theorems concerning the fractional positive continuous-time linear systems and their stability are recalled. The main result of the paper is given in section 3 where it is shown that the matrix A of the stable fractional positive system has not eigenvalues in the part of stability region located in the right half complex plane and the necessary and sufficient stability conditions are established. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. PRELIMINARIES

Consider the continuous-time linear system

$$ {}_0D_t^{\alpha}x(t)=Ax(t)\,,\ 0<\alpha<1 \tag{2.1}$$

where $x(t)\in\Re^n$ is the state vector and $A\in\Re^{n\times m}$,

$$ {}_0D_t^{\alpha}x(t)=\frac{d^{\alpha}x(t)}{dt^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}}d\tau,\ \ \dot{x}(\tau)=\frac{dx(\tau)}{d\tau} \tag{2.2}$$

is the Caputo definition of $\alpha\in\Re$ order derivative and

$$\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt \tag{2.3}$$

is the Euler gamma function.

The fractional system (2.1) will be called (internally) positive if $x(t)\in\Re_+^n$, $t\geq 0$ for any initial conditions $x(0)=x_0\in\Re_+^n$.

Theorem 2.1. (Kaczorek, 2011b) The fractional system (2.1) is positive if and only if

$$A\in M_n \tag{2.4}$$

where M_n is the set of $n\times n$ Metzler matrices.

Theorem 2.2. (Kaczorek, 2011b) The solution of equation (2.1) with initial conditions $x_0\in\Re^n$ is given by

$$x(t)=\Phi_0(t)x_0 \tag{2.5}$$

where

$$\Phi_0(t)=E_{\alpha}(At^{\alpha})=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{2.6}$$

and $E_{\alpha}(At^{\alpha})$ is the Mittage-Leffler matrix function.

The fractional positive system (2.1) will be called asymptotically stable (shortly stable) if

$$\lim_{t\to\infty}\Phi_0(t)x_0=0\ \text{ for all }\ x_0\in\Re_+^n \tag{2.7}$$

The characteristic polynomial of the matrix A of the fractional system (2.1) has the form

$$p_A(\lambda) = \det[I_n\lambda - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0, \quad \lambda = s^\alpha \tag{2.8}$$

Theorem 2.3. (Kaczorek, 2011b) The fractional system (2.1) is stable if and only if

$$\min_i |\arg \lambda_i| > \alpha\frac{\pi}{2} \tag{2.9}$$

where λ_i is the i-th eigenvalue of the matrix A.

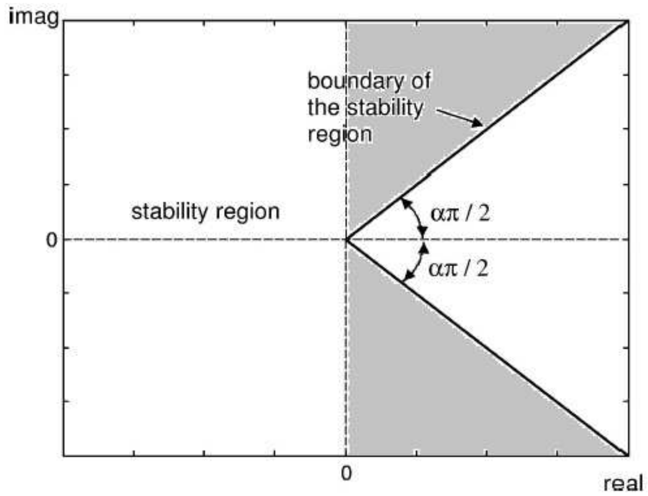

Fig. 2.1. Stability region

Theorem 2.4. (Kaczorek, 2011b) The fractional system (2.1) is unstable if at least one diagonal entry of the matrix A is positive.

3. MAIN RESULT

In this section necessary and sufficient stability conditions of the fractional positive system (2.1) will be established.

Theorem 3.1. The fractional positive system (2.1) for $0 < \alpha < 1$ is (asymptotically) stable if and only if

$$\mathrm{Re}\,\lambda_i < 0 \text{ for } i = 1,\ldots,n \tag{3.1}$$

Proof. By Theorem 2.1 the fractional system (2.1) is positive if and only if A is a Metzler matrix. It is well-known (Farina and Rinaldi, 2000; Mitkowski, 2008) that the dominant eigenvalue $\lambda_d = \lambda_1$ i.e.

$$\lambda_d > \mathrm{Re}\,\lambda_i \text{ for } i = 2,\ldots,n \tag{3.2}$$

of the Metzler matrix A is real. Therefore, the fractional positive system (2.1) is stable if and only if the condition (3.1) is satisfied.

From Theorem 3.1 we have the following important corollary.

Corollary 3.1. The matrix A of stable fractional positive system (2.1) has not eigenvalues in the part of stability region located in the right half complex plane (dark region on Fig. 2.1).

Let $A = [a_{ij}] \in \Re^{n\times m}$ be a Metzler matrix with negative diagonal entries ($a_{ii} < 0,\ i = 1,\ldots,n$).

Let define

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1,n}^{(0)} \\ \vdots & \ldots & \vdots \\ a_{n,1}^{(0)} & \ldots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} a_{11}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & A_{n-1}^{(0)} \end{bmatrix},$$

$$A_{n-1}^{(0)} = \begin{bmatrix} a_{22}^{(0)} & \ldots & a_{2,n}^{(0)} \\ \vdots & \ldots & \vdots \\ a_{n,2}^{(0)} & \ldots & a_{n,n}^{(0)} \end{bmatrix}, \tag{3.3a}$$

$$b_{n-1}^{(0)} = [a_{12}^{(0)} \quad \ldots \quad a_{1,n}^{(0)}], \quad c_{n-1}^{(0)} = \begin{bmatrix} a_{21}^{(0)} \\ \vdots \\ a_{n,1}^{(0)} \end{bmatrix}$$

and

$$A_{n-k}^{(k)} = A_{n-k}^{(k-1)} - \frac{c_{n-k}^{(k-1)} b_{n-k}^{(k-1)}}{a_{k+1,k+1}^{(k-1)}}$$

$$= \begin{bmatrix} a_{k+1,k+1}^{(k)} & \ldots & a_{k+1,n}^{(k)} \\ \vdots & \ldots & \vdots \\ a_{n,k+1}^{(k)} & \ldots & a_{n,n}^{(k)} \end{bmatrix} = \begin{bmatrix} a_{k+1,k+1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & A_{n-k-1}^{(k)} \end{bmatrix},$$

$$A_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+2}^{(k)} & \ldots & a_{k+2,n}^{(k)} \\ \vdots & \ldots & \vdots \\ a_{n,k+2}^{(k)} & \ldots & a_{n,n}^{(k)} \end{bmatrix}, \tag{3.3b}$$

$$b_{n-k-1}^{(k)} = [a_{k+1,k+2}^{(k)} \quad \ldots \quad a_{k+1,n}^{(k)}], \quad c_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+1}^{(k)} \\ \vdots \\ a_{n,k+1}^{(k)} \end{bmatrix}$$

for $k = 1,\ldots,n-1$.

Let us denote by $R[i + j \times c]$ the following elementary column operation on the matrix A: addition to the i-th column the j-th column multiplied by a scalar c. It is well-known that using these elementary operations we may reduce the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,n} \\ a_{21} & a_{22} & \ldots & a_{2,n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} \end{bmatrix} \tag{3.4}$$

to the lower triangular form

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & 0 & \ldots & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \ldots & \tilde{a}_{n,n} \end{bmatrix}. \tag{3.5}$$

To check the stability of the fractional positive system (2.1) the following theorem is recommended.

Theorem 3.2. The fractional positive linear system (2.1) for $0 < \alpha < 1$ is (asymptotically) stable if and only if one of the equivalent conditions is satisfied:

1. All principal minors $\Delta_i,\ i = i = 1,\ldots,n$ of the matrix $-A = [-a_{ij}]$ are positive, i.e.

$$\Delta_1 = -a_{11} > 0,\quad \Delta_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0,\quad \vdots \quad \Delta_n = \det[-A] > 0 \tag{3.6}$$

2. The diagonal entries of the matrices (3.3)

$$A_{n-k}^{(k)} \text{ for } k = 1,\ldots,n-1 \tag{3.7}$$

   are negative,
3. The diagonal entries of the lower triangular matrix (3.5) are negative, i.e.

$$\tilde{a}_{kk} < 0 \text{ for } k = 1,\ldots,n \tag{3.8}$$

Proof is given in Kaczorek (2011a).

Example 3.1. Consider the fractional system (2.1) with the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -a \end{bmatrix}. \tag{3.9}$$

Find the values of a for which the fractional positive system is stable. The fractional system is positive for all values of the entry a.

Using the conditions (3.6) for (3.9) we obtain

$$\Delta_1 = -a_{11} = 2 > 0,\quad \Delta_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} = 6 > 0 \tag{3.10a}$$

and

$$\det[-A] = \begin{vmatrix} 2 & 0 & -1 \\ -1 & 3 & 0 \\ -2 & -1 & a \end{vmatrix} = 6a - 7 > 0 \text{ for } a > 7/6. \tag{3.10b}$$

Therefore, the fractional positive system is stable for $a > 7/6$.

Using the conditions (3.7) for (3.9) we obtain

$$A_2^{(1)} = \begin{bmatrix} -3 & 0 \\ 1 & -a \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}[0 \quad 1] = \begin{bmatrix} -3 & 0.5 \\ 1 & 1-a \end{bmatrix}$$
$$A_1^{(2)} = [1-a] + \frac{1 \cdot 0.5}{3} = \frac{7}{6} - a. \tag{3.11}$$

The condition 2) of Theorem 3.2 is satisfied and the fractional positive system is stable for $a > 7/6$.

Similarly, using the elementary column operations to the matrix (3.9) we obtain

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -a \end{bmatrix} \xrightarrow{R[3+1\times 0.5]} \begin{bmatrix} -2 & 0 & 0 \\ 1 & -3 & 0.5 \\ 2 & 1 & 1-a \end{bmatrix}$$
$$\xrightarrow{R\left[3+2\times\frac{1}{6}\right]} \begin{bmatrix} -2 & 0 & 0 \\ 1 & -3 & 0 \\ 2 & 1 & \frac{7}{6}-a \end{bmatrix}. \tag{3.12}$$

The condition 3) of Theorem 3.2 is also satisfied and the fractional positive system is asymptotically stable for $a > 7/6$.

The characteristic polynomial of the matrix (3.9)

$$p_A(\lambda) = \det[I_3\lambda - A] = \begin{vmatrix} \lambda+2 & 0 & -1 \\ -1 & \lambda+3 & 0 \\ -2 & -1 & \lambda+a \end{vmatrix}$$
$$= \lambda^3 + (5+a)\lambda^2 + (5a+4)\lambda + 6a - 7 \tag{3.13}$$

has all positive coefficients if and only if $a > 7/6$. This also confirm Kaczorek (2011a) that the fractional positive system is stable if $a > 7/6$.

4. CONCLUDING REMARKS

Necessary and sufficient conditions for the asymptotic stability of fractional positive continuous-time linear systems have been established (Theorem 3.1). It has been shown (Corollary 3.1) that the matrix A of the stable fractional positive system has not eigenvalues in the part of stability region (Fig. 2.1) located in the right half of the complex plane. These considerations can be extended to positive fractional continuous-time linear systems with delays.

REFERENCES

1. **Busłowicz M.** (2010), Robust stability of positive discrete-time linear systems of fractional order, *Bull. Pol. Acad. Sci. Techn.*, vol. 58, no. 4, 567-572.
2. **Busłowicz M. and Kaczorek T.** (2009), Simple conditions for practical stability of positive fractional discrete-time lineart stystems, *Int. J. Appl. Math. Comput. Sci.*, Vol. 19, no. 2, 263-169.
3. **Farina L. and Rinaldi S.** (2000), *Positive Linear Systems; Theory and Applications*, J. Wiley, New York.
4. **Kaczorek T.** (2002), *Positive 1D and 2D Systems*, Springer Verlag, London.
5. **Kaczorek T.** (2010), Stability and stabilization of positive fractional linear systems by state-feedbacks, *Bull. Pol. Acad. Sci. Techn.*, vol. 58, no. 4, 517-554.
6. **Kaczorek T.** (2011a), New stability tests of positive standard and fractional linear systems, *Submitted to Circuits and Systems*.
7. **Kaczorek T.** (2011b), *Selected Problems of Fractional System Theory*, Springer Verlag, Berlin.
8. **Mitkowski W.** (2008), Dynamical properties of Metzler systems, *Bull. Pol. Acad. Sci. Techn.*, Vol. 56, no. 4, 309-312.
9. **Narendra K.S. and Shorten R.** (2010), Hurwitz Stability of Metzler Matrices, *IEEE Trans. Autom. Contr.*, Vol. 55, no. 6 June, 1484-1487.

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