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SURFACE LOCALIZED HEAT TRANSFER IN PERIODIC COMPOSITES

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Abstract: A characteristic feature of the description of physical phenomena formulated by an appropriate boundary or initial-boundary value problem and occurring in microstructured materials is the investigation of the unknown field in the form of decomposition referred to as micro-macro hypothesis. The first term of this decomposition is usually the integral average of the unknown physical field. The second term is a certain disturbance imposed on the first term and is represented in the form of a finite or infinite number of singleton fluctuations. Mentioned expansion is usually referred to as a two-scale expansion of the unknown physical field. In the paper, we purpose to apply two-scale expansion in the form of a certain Fourier series as a result of an applying Surface Localization of the unknown field. The considerations are illustrated by two examples, which results in analytical approximated solutions to the Effective Heat Conduction Problem for periodic composites, including the full dependence on the microstructure length parameter.

Key words: Heat transfer, periodic composites, homogenized model

1. INTRODUCTION

The procedure that is proposed in this paper can be treated as a certain variant of the Tolerance Averaging Technique (*TAT*) (Woźniak and Wierzbicki, 2000; Jędrysiak, 2010; Michalak, 2010; Ostrowski, 2017) or as an alternative purpose to the forming of two-scale expansions (Ariault, 1983; Bensoussan et al., 2011), proposed as a micro-macro representation of the unknown physical fields (Woźniak and Wierzbicki, 2000; Jędrysiak, 2010; Michalak, 2010; Ostrowski, 2017). Obtained in the subsequent considerations equivalent reformulation of Heat Transfer Equation (*HTE*) uses Fourier expansion as a representation of the temperature field and consists of: (1) a single equation for average temperature as for the first term of the mentioned expansion, (2) infinite number of equations for Fourier coefficients (amplitudes), and 3) finite number of tolerance amplitudes. Fourier basis taken into account in the proposed approach includes the possible changes of the composite periodicity along directions perpendicular to the periodicity directions. Hence, we can also deal with the FGM-type periodicity (Woźniak et al., 2002), similar to the twin but approximately original tolerance description of composite behaviours developed in Woźniak et. al. (2002), and continuators (Woźniak and Wierzbicki, 2000; Ostrowski, 2017). The sum of Fourier fluctuating terms (without the first term equal to the average temperature) can be interpreted as the analytical formula for the error made in using the approximate solutions of *HTE* proposed in *TAT* approach.

The starting point of considerations is the known parabolic heat transfer equation:

$$\nabla^T(K\nabla\theta) - c\dot{\theta} = b \qquad (1)$$

in which, the region $\Omega \subset R^D$, $2 \le D \le 3$, occupied by the composite is restricted to the form:

$$\Omega = \Omega_d \times \Omega_{D-d} \qquad (2)$$

in which, 1° $\Omega_d = (0, L)$, $\Omega_{D-d} = (0,\delta_1) \times (0,\delta_2)$ while $(d, D) = (1,3)$, 2° $\Omega_d = (0, L_1) \times (0, L_2)$, $\Omega_{D-d} = (0,\delta)$ while $(d, D) = (2,3)$, and 3° $\Omega_d = (0, L)$, $\Omega_{D-d} = (0,\delta)$ while $(d, D) = (1,2)$ for $L_1, L_2, L, \delta_1, \delta_2, \delta > 0$. In (2), $\theta = \theta(y, z, t)$, $y \in \Omega_d \subset R^d$, $z \in \Omega_{D-d} \subset R^{D-d}$, $t \ge 0$, denotes the temperature field, c is a specific heat and k is the heat conductivity constant matrix. Moreover, $\nabla \equiv \nabla_d + \nabla_{D-d}$ for $\nabla_d \equiv [\partial/\partial y^1, \dots, \partial/\partial y^d, 0, \dots, 0]^T$ with zeros placed in the last $D - d$ positions and $\nabla_{D-d} \equiv [0, \dots, 0, \partial/\partial z^1, \dots, \partial/\partial z^{D-d}]^T$ with zeros placed in the first d positions. Both fields $c = c(\cdot)$ and $k = k(\cdot)$ take S values c^1, ..., c^S and k^1, ..., k^S, respectively, do not depend on the temperature field θ and are restricted to Ω_d of a certain periodic field defined in R^d. Hence, considerations of the paper are restricted to Δ-periodic composites. Diameter $diam(\Delta)$ of repetitive cell is not necessarily small where compared to the characteristic length dimension L of the region Ω. With dimensionless scale parameter $\lambda = diam(\Delta)/L$, we will control the analysed equations in the subsequent considerations. The Δ- periodicity of the composite means that there exists σ-tuple $(\mathbf{v}^1, \dots, \mathbf{v}^d)$ of independent vectors $\mathbf{v}^1, \dots, \mathbf{v}^d \in R^d$ determining d directions of periodicity such that: (*i*) points $x + k_1\mathbf{v}_1 + \dots + k_d\mathbf{v}_d$, $-0.5 < k_1$, $k_d < 0.5$, cover for the interior of the cell $\Delta(x)$, (*ii*) $\Delta = \Delta(x_0)$ for fixed $x_0 \in R^3$ and (*iii*) $c(x + \mathbf{v}) = c(x)$, $K(x + \mathbf{v}) = K(x)$ for an arbitrary $\mathbf{v} \in \{\mathbf{v}_1, \dots, \mathbf{v}_d\}$, $x \in R^3$. The averaging $\langle f\rangle(x)$, $x \equiv (y, z)$, of an arbitrary integrable field f is defined by :

$$\langle f\rangle(x) = \frac{1}{|\Delta|}\int_\Delta f(\xi)d\xi \qquad (3)$$

and is a constant field provided that f is Δ-periodic.

2. REFORMULATION PROCEDURE

The investigations are based on the two fundamental assumptions. The first modelling assumption is a certain extension of the micro-macro hypothesis introduced framework of the tolerance averaging technique (Ariault, 1983; Bensoussan et al., 2011; Woźniak and Wierzbicki, 2000; Jędrysiak, 2010; Michalak, 2010; Ostrowski, 2017). In accordance with that hypothesis, the temperature field θ can be approximated with an acceptable accuracy by formula:

$$\theta_M(z) = \vartheta(z) + h^A(x)\psi_A(z) \tag{4}$$

in which, the slowly varying fields $\vartheta(\cdot)$ and $\psi_A(\cdot)$ are referred to as tolerance averaging of temperature field and amplitude fluctuations fields, respectively. Here and in the sequel, the summation convention holds with respect to indices $A = 1,\dots,N$. Symbols h^A, $A = 1,\dots,N$, used in (5) denote tolerance shape functions that should be periodic and satisfy conditions:

$$h^A \in o(\lambda), \lambda\nabla_y h^A \in o(\lambda), \langle ch^A\rangle = 0, \langle Kh^A\rangle = 0 \tag{5}$$

Usually, RHS of (4) is called Micro-Macro Decomposition of the temperature field. For particulars, the reader is referred to (Bensoussan et al., 2011; Woźniak and Wierzbicki, 2000; Jędrysiak, 2010). We interpret in (5), $\theta_{long} = \vartheta$ and $\theta_{short} = h^A(x)\psi_A(z)$. The tolerance-micro macro hypothesis can be formulated in the form:

Micro-Macro Hypothesis. The residual part of the temperature field θ_{res} being the difference between the temperature field θ and its tolerance part θ_M given by (4) can be treated as zero, $\theta_{res} \equiv \theta - \theta_M \approx 0$, that is, it vanish with an acceptable 'tolerance approximation'. The tolerance temperature part θ_M is debarked from the temperature field θ by the micro-macro hypothesis as an approximation of this field leading to the equation for the average temperature controlled by the finite number of fluctuation amplitudes $\psi_A(\cdot)$. We intend to supplement this micro-macro approximation to the complete temperature field θ interpreting decomposition:

$$\theta \equiv \theta_M + \theta_{res} \tag{6}$$

as a temperature field representation in with θ_{res} adding as the error made, while micro-macro decomposition (6) is used as tolerance approximation of the temperature field.

Taking into account the intention of adapting the idea implemented in the theory of signals, where we are dealing with the 'overlap' of many signals controlled by various parameters, we will try to impose onto decomposition (6) the interpretation dictated by the modified micro-macro hypothesis.

Modified Micro-Macro Hypothesis. The composite temperature field θ awards LS-decomposition onto the sum:

$$\theta \equiv \theta_L + \theta_S \tag{7}$$

of the long-wave part θ_L (L-part) and short-wave part θ_S (S-part), both sufficiently regular, which determine the disappearing heat flux vector component by:

$$(q_S)_n \equiv k(\nabla\theta_S)_n = 0 \tag{8}$$

normal to Γ, and hence, corresponding to:

$$\theta_S(y,z,t) \equiv \theta(y,z,t) - \theta_L(y,z,t) = a_p(z,t)\varphi^p(y,z) \tag{9}$$

as a certain orthogonal Fourier expansion representation of θ_S independent on thermal, material and geometrical composite properties. In restriction (8), allowing the use of the Fourier series in the unit vector field $n = n(x)$ is normal to discontinuity surfaces Γ in regular points x placed on Γ. Moreover, in (9), summation convention holds with respect to positive integer p. Moreover, decomposition (6) has been implemented in various ways. Modified micro-macro hypothesis will be supplemented with two remarks.

Remark. 1. If the orthogonality of Δ-periodic Fourier basis $\varphi^p(x)$, $p = 1,2,\dots$, is related to the scalar product $f_1 \circ f_2 = \langle f_1 f_2\rangle = \sum_{s=1}^{S} \eta_s \langle f_1 f_2\rangle_s$ using the `averaged values $\langle f_1 f_2\rangle_s$ taken over materially homogeneous parts Δ_s of the repetitive cell $\Delta \subset R^d$, then Fourier basis $\varphi^p(x)$ can be treated as independent on the material structure of the composite.

Remark. 2. Tolerance temperature approximation (4) is a certain temperature L-part θ_L provided that corresponding L-part $(q_M)_n \equiv n^T K\nabla\theta_M$ of heat flux normal component $(q)_n \equiv n^T K\nabla\theta$ is continuous on Γ. In this case, expansion (9) is equal to the error made under using $\theta_L = \theta_M$ as an approximation of θ. Moreover, the expansion:

$$\theta = \vartheta + \lambda[g^A\psi_A + a_p(z,t)\phi^p(y,z)] \tag{10}$$

is a certain temperature representation formed for $h^A(x,t) \equiv \lambda g^A(\lambda^{-1}x)$ and $\varphi^p(x,t) \equiv \lambda\phi^p(\lambda^{-1}x)$ and for:

$$\vartheta = a_0 + \theta_{res} - \lambda g^A\psi_A \tag{10}$$

together with two additional conditions for:

$$\begin{aligned}&\langle c\phi^p\rangle = 0, \langle k\phi^p\rangle = 0, p = 1,2,\dots,\\&\langle cg^A\rangle = 0, \langle kg^A\rangle = 0, A = 1,2,\dots,N.\end{aligned} \tag{11}$$

formulated under *Remark* 1.

We are to superimpose on the LS-decomposition a special interpretation in the framework, of which the composite behaviours being a direct consequence of the occurrence of the material discontinuity surfaces is described exclusively by the second term θ_S of $\theta \equiv \theta_L + \theta_S$ in (7) debarked from θ as supported on the ε-ribbon surrounding surfaces of material discontinuities of a composite, while the part θ_L of $\theta \equiv \theta_L + \theta_S$ does not notice the presence of a heterogeneous composite structure. Hence, the mentioned decomposition includes the natural decomposition of θ on a long-wave and a short-wave parts taken with respect to λ and localized inside and outside of the thin ε-ribbon surrounding surfaces of material discontinuities of a composite, respectively. Thus, decomposition $\theta \equiv \theta_L + \theta_S$ provides the ability to perform tolerance modeling procedure with respect to the field $u = \langle\vartheta\rangle$ as the average temperature field, and to the fields $\psi_A(\cdot)$ and $a_p(\cdot)$ as fluctuation amplitudes referred to as the tolerance and Fourier amplitudes, respectively, and also in relation to the new parameter ε as well as with respect to the small parameter λ. The aim of the paper is to overview the two-scale expansion (10) under its ε-asymptotic with respect to $\varepsilon \to 0$. As a result, we are to obtain an equivalent reformulation of *HTE* and then discuss the possibility to develope on this way the Effective Heat Conduction Equation in the form of a single equation for average temperature.

In order to realize the aim of the paper, we are to show that the following hypothesis holds.

Locality Hypothesis. The L-part θ_L of the temperature field can be supported on the ε-ribbon $o_\varepsilon(\Gamma)$ surrounding the discontinuity surfaces Γ, that is, $\theta_L(y,z,t) \neq 0$ for $(y,z) \in o_\varepsilon(\Gamma)$ and $\theta_L(y,z,t) = 0$ for $(y,z) \in \Omega\backslash o_\varepsilon(\Gamma)$ with the additional restriction $\nabla\theta_L(y,z,t) = 0$ satisfied for $(y,z) \in \Gamma \cup \partial o_\varepsilon(\Gamma)$ while.

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Locality hypothesis means that limit passage:

$$\theta_L = (\theta_L)_\varepsilon \to u \tag{12}$$

can be properly realized under $\varepsilon \to 0$. Formulating conditions sufficient for the average temperature u to coincide with its integral counterpart $u = \langle\theta\rangle$ is an open mathematical problem. This remark can be treated as a certain comment to the suggestions to compare effective modulus obtained as results of various methods of temperature averaging.

3. SURFACE LOCALIZATION PROCEDURE

Both L-part and S-part of the temperature θ depend on the parameter ε. In formal representation of the temperature L-part in the form of micro-macro decomposition, we are to adjust as the form of difference $\langle\theta_L\rangle(y,z,t) - \theta_L(y,z,t)$ to understanding its limit passage behaviour while $\varepsilon \to 0$. To this end, denoted by $\pi_\Gamma(y,z)$, the orthogonal projection of (y,z) onto Γ (well defined if (y,z) is placed sufficiently close to Γ). Let symbols n_A, $A = 1,\dots,S$, form the sequence of unit vector fields normal to $\partial\Delta_A \subset \Gamma$ and directed to the interior of Δ_A. Hence, $n_A(y,z) = -n_B(y,z)$ for $(y,z) \in \partial\Delta_A \cap \partial\Delta_B$ provided that Δ_A meets Δ_B along the fragment $\partial\Delta_A \cap \partial\Delta_B$ of Γ. Denoted by $\pi_S \subset \{(A,B): A,B = 1,\dots,S, A \neq B\}$, the set of all pairs (A,B) for which regions Δ_A and Δ_B are in contact with the surface panel $\partial\Delta_A \cap \partial\Delta_B \neq \emptyset$. Bearing in mind that $\langle\theta_L\rangle = \langle\theta\rangle$ is a differentiable function of ε, we arrive at:

$$\theta(y,z,t) = \langle\theta_L\rangle(z,t) + \frac{d\langle\theta_L\rangle}{d\varepsilon}\varepsilon + \theta_S + o(\varepsilon) \tag{13}$$

for sufficiently small $\varepsilon > 0$. Using decomposition $\frac{\partial}{\partial n_A} = v_A^y\nabla_y + v_A^z\nabla_z$ with respect to the orthogonal directions of y and z variables and $n_A = v_A^y n_y^A + v_A^z n_z^A$ as well as denotations:

$$\begin{aligned}\psi_{(A,B)}^{(y)}(y,z,t) &\equiv \langle(\nabla_y\theta_L)_A\rangle_{\partial\Delta_A\cap\partial\Delta_B}(y,z,t),\\ \psi_{(A,B)}^{(y)}(y,z,t) &\equiv \langle(\nabla_z\theta_L)_A\rangle_{\partial\Delta_A\cap\partial\Delta_B}(y,z,t),\end{aligned} \tag{14}$$

for tolerance amplitudes and:

$$\begin{aligned}&h_{(y)}^{(A,B)}(y,z)\\ &= -\eta_A(y,z)v_A^y(\pi_\Gamma(y,z))\frac{[\![k]\!]_{(A,B)}}{k_B}\frac{|\partial\Delta_A\cap\partial\Delta_B|}{|\partial\Delta_A|},\\ &h_{(z)}^{(A,B)}(y,z)\\ &= -\eta_A(y,z)v_A^z(\pi_\Gamma(y,z))\frac{[\![k]\!]_{(A,B)}}{k_B}\frac{|\partial\Delta_A\cap\partial\Delta_B|}{|\partial\Delta_A|},\end{aligned} \tag{15}$$

for tolerance shape functions in which $[\![k]\!]_{(A,B)} \equiv k_A - k_B$, we arrive at:

$$\begin{aligned}&\langle\theta_L\rangle(z,t) - \theta_L(y,z,t) =\\ &= \textstyle\sum_{A,B\in\pi_S}[h_{(y)}^{(A,B)}(y,z,t)\psi_{(A,B)}^{(y)}(z,t) +\\ &+h_{(y)}^{(A,B)}(y,z,t)\psi_{(A,B)}^{(y)}(z,t)] + o(\varepsilon)\end{aligned} \tag{16}$$

Hence (15) becomes Δ-periodic functions and (16) loses dependence of $\psi_{(A,B)}^{(y)}$ and $\psi_{(A,B)}^{(y)}$ on y-variable, while $\varepsilon \to 0$. Moreover, $v_A^z = 0$ for homogeneous periodicity. Cell distribution will be restricted to that introduced by the following:

Cell distribution hypothesis. For any periodic composite there exists cell distribution $\Delta(y,z)$, $(y,z) \in \Omega$, for which saturations η_A do not depend on y-variable.

Under rescaling:

$$g_{(\gamma)}^{\omega}(y,z) \equiv \lambda h_{(\gamma)}^{\omega}(\lambda^{-1}y,z) \quad for \quad \gamma = y,z \tag{17}$$

condition $\lim_{\varepsilon\searrow 0}\langle v_A^y(\pi_\Gamma(y,z))\rangle = 0$ reduces expansion (16) to:

$$\begin{aligned}\theta(y,z,t) &= \langle\theta_L\rangle(z,t) + \lambda[g_{(z)}^{\omega}(y,z)\psi_\omega^{(z)}(z,t) +\\ &+a_p(z,t)\phi^p(y,z)] + o(\varepsilon)\end{aligned} \tag{18}$$

In (18) summation over $\omega \equiv (A,B) \in \pi_S$ and over positive integer p holds. Bearing in mind that:

$$\vartheta = a_0 + \theta_{res} - \lambda g^A\psi_A \tag{19}$$

together with two additional conditions:

$$\begin{aligned}&\langle c\phi^p\rangle = 0, \langle k\phi^p\rangle = 0, p = 1,2,\dots,\\ &\langle cg^A\rangle = 0, \langle kg^A\rangle = 0, A = 1,2,\dots,N.\end{aligned} \tag{20}$$

we conclude that limit passage:

$$\theta_L = (\theta_L)_\varepsilon \to u \tag{21}$$

is properly realized under $\varepsilon \to 0$. Denote by H quadratic matrix with components $\langle k\nabla^T{}_y g_{(y)}^{v}, k\nabla_y g_{(z)}^{\mu}\rangle$. By applying orthogonalization procedure, one can arrive at the Surface Localized Heat Transfer Model equations (Kula, 2015; Kula and Wierzbicki, 2015, Woźniak et al., 2002; Wodzyński et al., 2018):

$$\begin{aligned}&\langle c\rangle\dot{u} - \nabla^T(\mathbb{k}_{surf}[\nabla u] + [k]_{surf}{}^p a_p) = -\langle b\rangle\\ &\lambda^2(\langle\phi^p c\phi^q\rangle\dot{a}_q - \nabla_z{}^T\langle\phi^p c\phi^q\rangle\nabla_z a_q) +\\ &+2\lambda s^{pq}\nabla_z a_q + \{k\}_{surf}{}^{pq}a_p = L_a^\lambda[u]\end{aligned} \tag{22}$$

In (22):

$$\begin{aligned}&\mathbb{k}_{surf} = \langle k\rangle - \langle k\nabla^T{}_y g_{(y)}^{v}, k\nabla_y g_{(z)}^{v}\rangle(H^{-1})_{v\mu}\begin{bmatrix}\langle\nabla_y{}^T g_{(y)}^{\mu}k\rangle\\ \langle\nabla_y{}^T g_{(z)}^{\mu}k\rangle\end{bmatrix},\\ &[k]_{surf} = k\langle\nabla^T\phi^p\rangle -\\ &-\langle k\nabla^T{}_y g_{(y)}^{v}, k\nabla_y g_{(z)}^{v}\rangle(H^{-1})_{v\mu}\begin{bmatrix}\langle\nabla_y{}^T g_{(y)}^{\mu}k\nabla_z\phi^q\rangle\\ \langle\nabla_y{}^T g_{(z)}^{\mu}k\nabla_z\phi^q\rangle\end{bmatrix},\\ &2s^{pq} = \langle\nabla_y{}^T\phi^p k\phi^q\rangle - \langle\nabla^T\phi^q k\phi^p\rangle,\\ &\{k\}^{pq} = \langle\nabla_y{}^T\phi^p k\nabla\phi^q\rangle,\end{aligned} \tag{23}$$

are used as additional denotations. Matrix coefficient $\mathbb{k}_{surf} = \mathbb{k}_{surf}(z)$ is referred to as Surface Localized part of Effective Conductivity Matrix. The homogenized part:

$$\begin{aligned}&\lambda^2(\langle\phi^p c\phi^q\rangle\dot{a}_q - \nabla_z{}^T\langle\phi^p c\phi^q\rangle\nabla_z a_q) +\\ &+2\lambda s_{surf}{}^{pq}\nabla_z a_q + \{k\}_{surf}{}^{pq}a_p = 0\end{aligned} \tag{24}$$

of $(22)_2$ describing uninhibited by external influences represented by the RHS part $L_a^\lambda[u]$ of (22) is usually considered as a model of the Boundary Effect Equation. The investigation of the reduction of Fourier amplitudes a_p from $(22)_2$ leads to the single equation for average temperature u referred to as the Effective Conductivity Equation. Imitating the appropriate procedure, the Effective Conductivity Equation without scale effect can be easily realized in the form:

$$\langle c\dot{u}\rangle - \nabla^T(\mathbb{k}_0[\nabla_z u] + \langle\phi^p\nabla_z{}^T(k\nabla_z u)\rangle) = -\langle b\rangle \tag{25}$$

from the asymptotic case of model equations:

$$\langle c\dot{u}\rangle - \nabla^T(\mathbb{k}_0{}^{eff}[\nabla_z u] + [k]_{surf}{}^p a_p) = -\langle b\rangle$$
$$\{k\}_{surf}{}^{pq} a_p = L_a^0[\nabla_z u] \tag{26}$$

for $\lambda \searrow 0$ and for $\mathbb{k}_0{}^{eff}$ given by:

$$\mathbb{k}_0{}^{eff}[\nabla_z u] = \mathbb{k}_{surf}[\nabla_z u] - [k]_{surf}{}^p \cdot$$
$$\cdot (\{k\}_{surf}{}^{-1})^{pq}\big(\langle \nabla_y{}^T \phi^q k\rangle \nabla_z u + \langle \phi^q \nabla_z{}^T(k\nabla_z u)\rangle\big) \tag{27}$$

As the benchmark problem, we are to propose a procedure of eliminating Fourier amplitudes from (22)2, which results the derivation of Effective Conductivity single equation for average temperature. The investigation will be restricted to the special case for $D = 2$ and $d = 1$.

4. PASSAGE TO THE EFFECTIVE CONDUCTIVITY

Let $\mathrm{D} = 2$, $\sigma = 1$. Hence, k is a symmetric 2×2 positive matrix and let $k = [k_{yy}, k_{yz}; k_{zy}, k_{zz}]$, $k_{yz} = k_{zy}$, and hence, we deal with two-dimensional periodic layer. Moreover, denote:

$$(L_a^\lambda[\tfrac{du}{dz}])^p \equiv \langle \tfrac{d\Phi^p}{dy} k_{yz} + \tfrac{d\Phi^p}{dz} k_{zz}\rangle \tfrac{du}{dz} - \lambda\langle \varphi^p b\rangle \tag{28}$$

We are to take a stationary case:

$$\tfrac{d}{dz}(\mathbb{k}_{surf}\tfrac{du}{dz} + [k]^p a_p) = \langle b\rangle$$
$$\langle \phi^p c \phi^q\rangle \tfrac{d^2 a_p}{dz^2} - 2\lambda s^{pq}\tfrac{da_q}{dz} - \{k\}^{pq} a_p = -L_a^\lambda[u] \tag{29}$$

of (22) being in this case the second order ordinary differential equations. Let $A_k{}^{pq} = \langle \phi^p c \phi^q\rangle$ be elements of quadratic matrix A. Moreover, let:

$$\Delta = s_{surf}{}^2 + A\{k\}_{surf},\ r_\pm = -A_k{}^{-1}(\sqrt{\Delta} \pm s_{surf}),$$
$$R \equiv r_+ - r_-, \tag{30}$$

where: square root is used here for positive quadratic matrix Δ and is equal to the unique positive matrix $\sqrt{\Delta}$ for which $\sqrt{\Delta}^2 = \Delta$ while $\frac{s_{surf}}{A_k} = s_{surf}\cdot A_k{}^{-1} = A_k{}^{-1}\cdot s_{surf}$ for any matrices alternating in multiplication. Similarly, $\frac{\sqrt{\Delta}}{A_k} = \sqrt{\Delta}\cdot A_k{}^{-1} = A_k{}^{-1}\cdot\sqrt{\Delta}$ for positive matrices Δ and R. Moreover, let

$$\ker_R(\omega) = R^{-1}\cdot \mathrm{sh}R\omega, \mathrm{coker}_R(\omega) = \mathrm{ch}R\omega \tag{31}$$

for $\omega \in \{(z-\delta)/\delta, z/\delta\}$, and consider $a \equiv [a_1, a_2, \ldots]^T = \sigma^h + \sigma^s$ as a solution to (29)2 (with respect to Fourier amplitudes a_p) satisfying $a(0) = a_0$ and $a(\frac{\delta}{\lambda}) = a_\delta$ as attached boundary conditions. Integral:

$$\tfrac{1}{2}\lambda A_k \sigma^s(z) = I_0 \equiv \int_0^z \ker(\tfrac{z-\xi}{\lambda}) L_a^\lambda[u](\xi)\, d\xi \tag{32}$$

is considered as a formula for an arbitrary but fixed solution σ^s of (29)2 and:

$$\sigma^h(\tfrac{z}{\lambda}) = \frac{\sinh R\frac{z}{\lambda}}{\sinh R\frac{\delta}{\lambda}}[a^\delta - \sigma^s(\tfrac{\delta}{\lambda})] - \frac{\sinh R\frac{z-\delta}{\lambda}}{\sinh R\frac{\delta}{\lambda}} a^0 \tag{33}$$

is a solution to homogeneous part:

$$\lambda^2\langle \phi^p c \phi^q\rangle \tfrac{d^2 a_p^h}{dz^2} - 2\lambda s^{pq}\tfrac{da_p^h}{dz} - \{k\}^{pq} a_p^h = 0 \tag{34}$$

of (29)2 satisfying $a_p^h(0) = a_0 - \sigma^s(0)$ and $a_p^h(\delta/\lambda) = a_\delta - \sigma^s(\delta/\lambda)$. Double integration:

$$I_0 = \int_0^z \ker(\tfrac{z-\xi}{\lambda}) L_a^\lambda[u](\xi)\, d\xi = \lambda R^{-1}\{L_a^\lambda[u](z) -$$
$$-\mathrm{coker}_R(\tfrac{z}{\lambda})L_a^\lambda[u](0)\} - \lambda R^{-1}\cdot\{\ker_R(\xi)\tfrac{dL_a^\lambda[u](0)}{dz} +$$
$$+ \int_0^z \ker_R(\tfrac{z-\xi}{\lambda})\tfrac{d^2 L_a^\lambda[u](\xi)}{dz^2} d\xi\} \tag{35}$$

realized under denotations:

$$X_{2k}(z,\xi) \equiv \ker_R(\tfrac{z-\xi}{\lambda})\tfrac{d^{2k}L_a^\lambda[u]}{dz^{2k}}(\xi)$$
$$Y_{2k}(z,\xi) \equiv \mathrm{coker}_R(\tfrac{z-\xi}{\lambda})\tfrac{d^{2k}L_a^\lambda[u]}{dz^{2k}}(\xi) \tag{36}$$

results formula:

$$\sigma^s(z) = \frac{A_k}{2}\cdot\sum_{k=0}^n \lambda^{2k}(R^{-1})^{2k-1}\{Y_{2k}(z,z) - Y_{2k}(z,0) -$$
$$-(\lambda R^{-1})[-X_{2k+1}(z,z) + X_{2k+1}(z,0)]\} +$$
$$-(-\lambda)^{2n+1}\frac{A_k}{2}\cdot(R^{-1})^{2k}\int_0^z X_{2k+2}(z,\xi)d\xi \tag{37}$$

Let $[k]^T \equiv [[k]^1, [k]^2, \ldots]$. Coefficients:

$$\mathbb{k}_{2k}(z) \equiv [k]^T\tfrac{A_K}{2}\cdot(R^{-1})^{2k-1}[k] =$$
$$-[k]^T\tfrac{A_K}{2}\cdot(R^{-1})^{2k-1}[Y_{2p}(z,z) - \lambda R^{-1}X_{2p+1}(z,z)] =$$
$$= [k]^T\{-\tfrac{A_K}{2}\cdot(R^{-1})^{2p-1}\cdot\mathrm{coker}_R(\tfrac{z-\xi}{\lambda})\tfrac{d^{2k}L_a^\lambda[u]}{dz^{2k}}(\xi) +$$
$$+\lambda R^{-1}\cdot\ker_R(\tfrac{z-\xi}{\lambda})\tfrac{d^{2k}L_a^\lambda[u]}{dz^{2k}}(\xi)\}|_{\xi=z}$$
$$\langle b\rangle_{2k}(z) \equiv -[k]^T\tfrac{A_K}{2}\cdot(R^{-1})^{2k-1}[Y_{2k}(z,0) -$$
$$-\lambda R^{-1}X_{2k+1}(z,0)] =$$
$$= [k]^T\{-\tfrac{A_K}{2}\cdot(R^{-1})^{2k-1}\cdot\mathrm{coker}_R(\tfrac{z-\xi}{\lambda})\tfrac{d^{2k}L_a^\lambda[u]}{dz^{2k}}(\xi) +$$
$$+\lambda R^{-1}\cdot\ker_R(\tfrac{z-\xi}{\lambda})\tfrac{d^{2k}L_a^\lambda[u]}{dz^{2k}}(\xi)\}|_{\xi=0} \tag{38}$$

are amplitudes of corrections:

$$\mathbb{k}^s(z) = \textstyle\sum_{k=1}^{+\infty}(-1)^{k-1}\lambda^{2k}\mathbb{k}_{2k}(z)$$
$$\langle b\rangle^s(z) \equiv \textstyle\sum_{k=1}^{+\infty}(-1)^{k-1}\lambda^{2k}\langle b\rangle_{2k}(z) \tag{39}$$

imposed on Effective Conductivity $\mathbb{k}^{eff}(z)$ and the average sources $\langle \mathrm{b}\rangle$ in formulas $\mathbb{k}_\lambda^{\mathrm{eff}}(z) \equiv \mathbb{k}^{\mathrm{eff}}(z) + \mathbb{k}^{\mathrm{s}}(z)$ as well as $\langle b\rangle_\lambda^{eff} \equiv \langle b\rangle + \mathbb{k}^{\mathrm{s}}(0) - \sigma^{hom}(z)$ for Effective Conductivity and Effective Sources, respectively, depends of λ. Formula:

$$res_{2n+1}[u](z) =$$
$$(-1)^{n+1}\lambda^{2n+1}\tfrac{A_k}{2}\cdot(R^{-1})^{2n}\int_0^z X_{2n+2}(z,\xi)d\xi \tag{40}$$

represents error made while $\mathbb{k}^s(z)$, $\mathbb{k}^{\mathrm{s}}(0)$ are replaced by:

$$\mathbb{k}_{(2N)}{}^s(z) = \textstyle\sum_{n=1}^{2N}(-1)^{n-1}\lambda^{2k}\mathbb{k}_{2k}(z)$$
$$\langle b\rangle_{(2N)}{}^s(z) \equiv \textstyle\sum_{n=1}^{2N}(-1)^{n-1}\lambda^{2k}\langle b\rangle_{2k}(z) \tag{41}$$

of (39). Modulus of (39) can be formally estimated by the supremum of modulus of the RHS side of (40) proportional to λ^{2n+1}. That is why, $\lambda < 1$ is a sufficient condition for convergence of RHS of (40) to zero, while $n \to \infty$. Finally, we obtain:

$$\tfrac{d}{dz}(\mathbb{k}_\lambda^{eff}(z)\tfrac{du}{dz}) = \langle b\rangle_\lambda^{eff} \tag{42}$$

instead of (29).

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5. FURTHER APPROXIMATIONS OF THE EFFECTIVE CONDUCTIVITY EQUATION

Effective Conductivity problem will be examined as a boundary value problem for $2N$-th order ordinary differential equation:

$$\frac{d}{dz}\{\mathbb{k}^{eff}(z)\frac{du}{dz}+\sum_{n=1}^{N}(-1)^{n+1}\lambda^{2n}\mathbb{k}_{2n}(z)\frac{d^{2n-1}u}{dz^{2n-1}}\}= \\ =\langle b\rangle^{eff}+\sum_{n=0}^{+\infty}(-1)^{n+1}\lambda^{2n}\langle b\rangle_{2n}(z) \quad (43)$$

in which:

$$\mathbb{k}_{(2N)}{}^{s}(z)=\sum_{n=1}^{2N}(-1)^{n-1}\lambda^{2n}\mathbb{k}_{2n}(z)\frac{d^{2n}u}{dz^{2n}} \quad (44)$$

The following uniqueness conditions:

$$\frac{d^{n-1}u}{dz^{n-1}}(0)=u_0^{n-1}, n=1,\dots,N, \quad (45)$$

will be attached. Formulated Cauchy problem is considered as $2N$th order approximation of Effective Conductivity problem.

Asymptotic approximation is considered as $N=0$. Now, mentioned Cauchy problem simplifies to:

$$\begin{cases}\frac{du}{dz}\{\mathbb{k}^{eff}(z)\frac{du}{dz}\}=\langle b\rangle^{eff}\\ u(0)=u_0,\quad u(\delta)=u_\delta\end{cases} \quad (46)$$

leads to the solution:

$$u=u_0+\int_0^z\frac{\mathrm{Q}(\xi)}{\mathbb{k}^{eff}(\xi)}d\xi, \mathrm{Q}=\mathrm{Q}_0+\int_0^z\langle b\rangle^{eff}(\xi)d\xi \quad (47)$$

for the average heat flux Q. For homogeneous periodicity, (47) reduces to the form:

$$u(z)=u_0+\frac{1}{\mathbb{k}^{eff}}(\mathrm{Q}_0+z\langle b\rangle^{eff}), \\ \mathrm{Q}_0=\mathbb{k}^{eff}(u_\delta-u_0)-\delta\langle b\rangle^{eff} \quad (48)$$

Approximate solutions to the considered Cauchy problem can be interpreted as subsequent overlaps on the unscaled average temperature approximation (48) valid for $N=0$. Denote:

$$f(z)\equiv\delta(0)+\int_{\varsigma=0}^{\varsigma=z}d\varsigma\frac{1}{\mathbb{k}^{eff}(\varsigma)}[\beta_0+\int_{\xi=0}^{\xi=\varsigma}\langle b\rangle_\lambda{}^{eff}(\xi)d\xi] \quad (49)$$

For:

$$\beta_0\equiv\mathbb{k}^{eff}(0)\frac{du(0)}{dz}+\sum_{n=0}^{N}(-1)^n\lambda^{2n+2}\mathbb{k}_{2n}(0)\frac{d^{2n+1}u(0)}{dz^{2n+1}}, \\ \delta_0\equiv u(0)+\sum_{n=0}^{N}(-1)^n\lambda^{2n+2}\frac{\mathbb{k}_{2n+2}(0)}{\mathbb{k}^{eff}(0)}\frac{d^{2n}u(0)}{dz^{2n}} \quad (50)$$

Double integration of (43) leads from (43) to the $2N$th order ordinary differential equation:

$$-u+\sum_{n=1}^{N}(-1)^n\lambda^{2n}\frac{\mathbb{k}_{2n+2}(z)}{\mathbb{k}^{eff}(z)}\frac{d^{2n}u}{dz^{2n}}=f(z) \quad (51)$$

with:

$$-1+\sum_{n=1}^{N}(-1)^n\lambda^{2n}\frac{\mathbb{k}_{2n+2}(z)}{\mathbb{k}^{eff}(z)}r^{2n}=f(z) \quad (52)$$

as a characteristic equation.

4th order approximation will be considered as $N=2$. In this case and for homogeneous periodicity related approximate solution to the Effective Conductivity Problem can be written as:

$$\begin{aligned}u(z)=&-\frac{k_{eff}}{\lambda^4k_6}[u(0)zf_s^+(z)*f_s^-(z)+\\&+u'(0)z^2f_s^+(z)*f_s^-(z)]+\\&-\frac{k_4}{\lambda^2k_6}[u(0)(f_c^+(z)*f_s^-(z)+\kappa f_s^+(z)*f_s^-(z))+\\&+u'(0)f_s^+(z)*f_s^-(z)+\\&+u^{(2)}(0)zf_s^+(z)*f_s^-(z)+u^{(3)}(0)z^2f_s^+(z)*f_s^-(z)]-\\&-[(A_{ch}(f_c^+(z)+f_c^-(z))+\\&+B_{sh}(f_s^+(z)-f_s^-(z)))u(0)+\\&+u^{(1)}(0)(f_c^+(z)*f_c^-(z)+\kappa^2f_s^+(z)*f_s^-(z))\\&+u^{(2)}(0)(f_c^+(z)*f_s^-(z)-\kappa f_s^+(z)*f_s^-(z))+\\&+u^{(3)}(0)f_s^+(z)*f_s^-(z)+\\&+u^{(4)}(0)zf_s^+(z)*f_s^-(z)+u^{(5)}(0)z^2f_s^+(z)*f_s^-(z)]-\\&-\frac{\langle b\rangle(z)}{\lambda^4k_6}f_s^+(z)*f_s^-(z)\end{aligned} \quad (53)$$

in which the convolution integral $f_1(z)*f_2(z)$ is used for:

$$f_c^+(z)=e^{\kappa\frac{z-\delta}{\lambda}}\cos\omega\frac{z}{\lambda}, f_s^+(z)=e^{\kappa\frac{z-\delta}{\lambda}}\sin\omega\frac{z}{\lambda}, \\ f_c^-(z)=e^{-\kappa\frac{z-\delta}{\lambda}}\cos\omega\frac{z}{\lambda}, f_s^-(z)=e^{-\kappa\frac{z-\delta}{\lambda}}\sin\omega\frac{z}{\lambda}. \quad (54)$$

and for not vanishing roots $\pm\kappa\pm\mathrm{i}\,\omega$, $i^2=-1$, $\kappa,\omega>0$, of algebraic equation (52) for $N=2$. Constant coefficients A_{ch} and B_{sh} in (53) should satisfy:

$$\begin{bmatrix}\frac{\omega+2\kappa}{(\omega+2\kappa)^2+\omega^2}+\frac{1}{2\omega} & \frac{1}{(\omega+2\kappa)^2+\omega^2}-\frac{1}{2\omega^2}\\ \frac{1}{2\omega}+\frac{\omega+2\kappa}{(\omega-2\kappa)^2+\omega^2} & \frac{1}{2\omega^2}-\frac{1}{(\omega-2\kappa)^2+\omega^2}\end{bmatrix}\cdot \\ \cdot\begin{bmatrix}A_{ch}\\B_{sh}\end{bmatrix}=\begin{bmatrix}\frac{\omega+2\kappa}{(\omega+2\kappa)^2+\omega^2}+\frac{1}{2\omega}\\ \frac{1}{2\omega}+\frac{\omega+2\kappa}{(\omega-2\kappa)^2+\omega^2}\end{bmatrix} \quad (55)$$

Hence, (53) are the basic formulas constituting the departure point for the prediction of the control form of the average temperature of the microstructural parameter. We intend to formulate an appropriate hypothesis for the case determined by $D=2$ and $d=1$ considered in this section.

6. FINAL HYPOTHESIS

In order to arrive at the main thesis of the paper, we will now generalize the results of the last section, resulting in the predicted distribution of the average temperature as a solution to the Effective Heat Conductivity problem controlled by the microstructural parameter λ. We formulate the following hypothesis:

Final Hypothesis. The sequence of subsequent solutions of $2N$th approximations of the Effective Conductivity problem can be written as a sum of terms of the form (54), where κ and ω run over all pairs corresponding to the collection of not vanishing roots $\pm\kappa\pm\omega\mathrm{i}$ of characteristic equation (52). Coefficients of this sum are uniquely determined by the initial values $u^{(n)}(0)$, $n=0,1,2,3,\ldots$.

However, limit formulas of this approximated solutions to the Effective Conductivity equation, developed while $\lambda\to0$, exist in a weak sense. It is also worth emphasizing that the interpretation of the first component of the Fourier expansion (19) as the average temperature is obtained as the integration effect of the infinite function series and thus uses as the assumption that the various limit operations which are applied are alternating.

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